A DEFORMABLE-RADIUS B-SPLINE METHOD FOR SHAPE-BASED INVERSE PROBLEMS, AS APPLIED TO ELECTRICAL IMPEDANCE TOMOGRAPHY

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ABSTRACT

Shape-based solutions have recently received attention for certain ill-posed inverse problems when the relevant constituent parameters can be modeled as isotropic and piecewise continuous or homogeneous. Their advantages include implicit imposition of relevant constraints and reduction in the number of unknowns, especially important for non-linear ill-posed problems. We introduce a method, based on modified B-splines and the Boundary Element Method (BEM), for shape-based inverse problems and apply it to current-injection Electrical Impedance Tomography (EIT) in the torso. We assume the general shape of piecewise constant inhomogeneities is known but their conductivities and their exact location and shape is not. We model the boundaries of the unknown inhomogeneities using bivariate spline basis shape functions and solve for the conductivities, locations, and boundaries. The performance of our method is illustrated via simulation in a realistic torso model.

1. INTRODUCTION

Electrical Impedance Tomography (EIT) is an imaging modality in which the relationship between currents and voltages on the surface of a volume is used to estimate the conductivity (or more generally, the impedance) map in the interior. In particular, we address here the problem of determining the conductivity inside the torso by injecting currents with an array of electrodes and measuring voltages with the same electrodes. This is of interest in many application areas, including monitoring of the mechanics of heart and lung function and electrocardiography [1].

EIT is a very badly posed inverse problem, and in a 3D volume requires far too many parameters to be estimated in a stable fashion if one wants to obtain good spatial resolution and good accuracy in conductivity estimates. In addition, the dependence of the conductivity on the measured quantities is non-linear. One approach to such problems that has been treated recently in a number of reports is to parameterize the geometry in a fashion that reduces the number of unknowns. In particular, in problems such as EIT of the torso volume, it is reasonable to model the volume as being composed of a relatively small number of piecewise-constant (or more generally, piecewise homogeneous) regions. In that case the unknowns are reduced to the location of the boundaries of these regions and one (or a few) parameter value(s) per region. Such approaches are often called "shape-based inverse solutions", since the unknown boundaries are typically modeled with some parametric shape functions. Among the approaches that have been applied are level sets , trigonometric series, and variations of closed-form analytic objects such as ellipsoids [2]. Spline-based methods have been applied in two-dimensions, including to EIT [3], but their extension to three-dimensions is notoriously difficult. (Due to space limitations we cannot cite a number of relevant reports of such methods, so we have selected only a few here.)

In such inverse problems, one also needs a model of the relationship between the unknown quantities and the measurements, known as a forward model. In EIT, this model is typically obtained by assuming the geometry and conductivities are known (so, in a shape-based model, assuming the region boundaries are known) and solving Laplace's equation in the interior of the torso (assuming the frequencies being used are small enough so that the quasi-static approximation holds, as is generally true in EIT imaging scenarios) with appropriate boundary conditions. One approach for solving Laplace's equation with the piecewise constant assumption that is particularly attractive for modeling and computational reasons is the Boundary Element Method (BEM). BEM requires that the surface of each region be represented by some sort of discrete mesh (typically via a triangulation); then the volume integrals derived from Laplace's equation are converted to surface integrals over these meshes. The difficulty in using BEM in the shape-based EIT problem described here is that the non-linearity of the EIT problem requires iterative solutions, but the mesh would need to be changed with every iteration. If one parameterizes the surface appropriately, however, one might be able to simplify this mesh re-generation problem sufficiently to make this approach computationally practical.

In this work, we report on the development of a method, based on a modified B-spline approach, that allows us to define a set of knots for the B-spline based on approximation assumptions about the objects which constitute the conductivity inhomogeneity in the interior. For instance, we would assume that we know that there is an inhomogeneity with the general shape of a heart inside the torso. Using this assumption, as we describe below, we set up a 3-stage optimization algorithm that allows us to iteratively estimate the location of this object, to find its external boundary, and to estimate its internal conductivity. We show a simulation example based on a realistic torso geometry and taking into consideration a single internal object, the heart.

In Sec. 2, we briefly describe our new 3D BEM-EIT forward

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model. We then present our method to model the internal surfaces using a bivariate spline shape function basis and propose an algorithm to solve the associated inverse problem. In Sec. 3, simulation results of applying this method on a 3D human torso model are shown. We discuss some details of the method presented and ideas for the future work in Sec. 4.

2. THE FORWARD AND INVERSE SOLUTIONS

In the forward problem, we assume the geometry and conductivities are known and we are looking for the relation between the current injected by electrodes placed on the outermost surface and the potential distribution on the same surface. By using the BEM-based method we introduced in [4], this relation can be represented as:

$$\Phi_1 = \mathbf{T}\Gamma_1 \tag{1}$$

where $\mathbf{T}$ is a dense matrix that directly relates Φ_1, the potential distribution on the outermost surface, to the current distribution on the same surface, Γ_1. $\mathbf{T}$ is a function of both the geometry and conductivities of all the objects.

The inverse problem, then, is to find a combination of parameters for which the difference between the measured potentials and the potentials predicted by the model are minimum, subject perhaps to some *a priori* constraints on those parameters. Here, these parameters represent both the conductivities and geometry of the internal objects.

We assume the shape of torso surface is known, but to reconstruct the shape of internal surfaces, first we need to represent them with as few parameters as possible. One approach is to mathematically describe the surfaces of the 3D inhomogeneities using some shape function basis.

2.1. Modeling the heart surface

Here, we briefly explain the method we developed to parameterize the heart surface using a bivariate spline shape function basis. The same method can be used for any convex surface.

In the first step we choose a few *control points* on the heart surface and unfold the heart surface by representing the surface in spherical coordinates, while assuming that the heart center is located at the origin. In the new coordinate system, we treat these control points as being distributed on a surface represented by $r = surf(\theta, \phi)$. Choosing the control points carefully, we make a gridded map of r on $-\pi/2 \leq \phi \leq \pi/2$ and $-\pi \leq \theta < \pi$. Note that given a set of (θ, ϕ) points, the shape of the heart is represented by the associated values of r. In the second step, we find a least square approximation to r by tensor product splines using the method introduced in [5]. In the third step, we unfold the fitted surface by going back to the Cartesian coordinates. Since after the second step, we have an analytical B-spline description of the heart surface, we can generate a triangular mesh for it with any density we like, which we then can use in our BEM model. In practice, we generate a triangular mesh of the unit sphere and then map its points to the B-spline heart surface to generate the heart mesh. Fig. 1 illustrates the proposed process.

2.2. Inverse solution algorithm

For convenience, we explain the method for the case when there is only one internal object, a heart with a constant (nonzero) conductivity, inside the torso. The extension to more internal piecewise constant objects is straightforward.

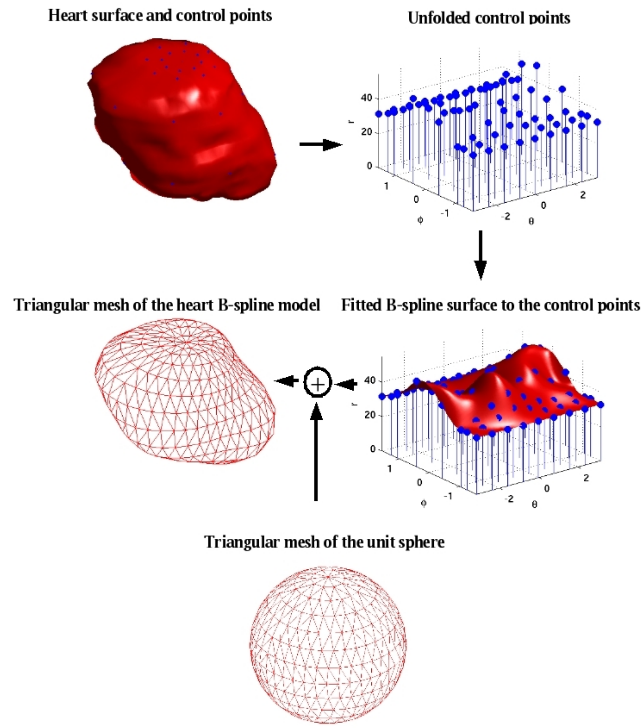

Fig. 1. The process of modeling the heart surface using bivariate splines and generating a triangular mesh for it.

In modeling the heart surface, since we *choose* the ϕ's and θ's of the control points, the only parameters of the heart surface we need to find in the inverse solution to reconstruct the heart shape, are the distances of the control points from the heart center, the r's. We also need to find the heart center to retrieve the relative position of the heart and torso. So we have been able to dramatically decrease the unknown parameters compared to representing the surface with a triangular mesh directly. In addition to the geometry, the piece-wise constant conductivities of different regions are also unknown and should be found in the inverse solution. So we propose a method to sequentially estimate these three different groups of unknowns.

We can think of the inverse problem as a nonlinear optimization problem to estimate a limited number of parameters. Thus our ability to find the conductivity values accurately depends upon the definition of the error function and the signal-to-noise ratio (SNR).

Since there is always some noise in the data, which can be a combination of measurement noise, modeling error, and numerical error, to prevent the inverse solution from giving us a shape which does not look like a real heart, we can constrain the heart shape to regularize the solution and then use the iterative algorithm shown in Fig. 2 to minimize the following quadratic error cost function

$$CF = \sum_{i=1}^{N} (E_i - V_i)^2 + \text{regularization term} \tag{2}$$

where E_i is the potential at electrode *i* predicted by the forward model with a given candidate inverse solution, V_i is the experi-

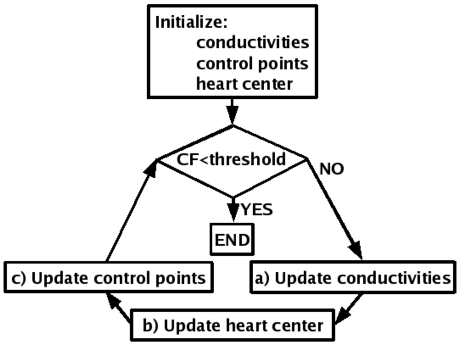

Fig. 2. The iterative algorithm for the inverse solution.

mentally measured potential at electrode *i*, and N is the number of electrodes.

The regularization term can be a constraint put either on the heart shape itself or on the control points. For example we can constrain the total area of the heart surface or its total volume or, since we know the reconstructed heart surface should be smooth, we can constrain the surface gradient or curvature. As an example of a constraint on the control points, we can use reference control points by averaging over such points based on real heart shapes using the method explained in section 2.1 and then use the regularization term to force the reconstructed heart surface to be *somewhat similar* to the reference heart model by setting the regularization term in (2) to be

$$\text{regularization term} = \lambda \sum_{i=1}^{M} (CP_i - CP_{ref,i})^2. \qquad (3)$$

Here CP_{ref} are the reference control points, CP are the desired control points, M is the number of control points and λ is a regularization parameter.

Each of the steps (a), (b) and (c) in the algorithm shown in Fig. 2 is itself a nonlinear optimization problem. To update the conductivities and the heart center in steps (a) and (b), we used the large-scale algorithm which is a subspace trust region method based on the interior-reflective Newton method [6]. To update the control points in step (c), we used the downhill simplex algorithm [7]. Each iteration in the algorithm in steps (a) and (b) involves the approximate solution of a large linear system using the method of preconditioned conjugate gradients (PCG). It is shown in [8] that for large problems, methods based on the CG iterations are more efficient than other methods like LU or Cholesky decomposition. Unlike the simplex method, which does not need any evaluation of the cost function's gradient (and hence, is quite easily implementable), the preconditioner computation used in the PCG part of the large-scale method needs the Jacobian matrix; therefore, we developed an algorithm to analytically compute the Jacobian matrix for the BEM solution, as required by this method. This significantly improves the speed of the solution comparing with using the Finite Difference Method (FDM) to compute the Jacobian matrix at each iteration or using the simplex algorithm to do the optimization. Using this algorithm, we can also put constraints on values of the heart center and conductivities by setting lower and upper bounds found from the literature.

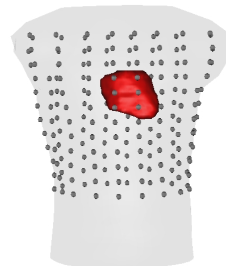

Fig. 3. The model of a male human torso with heart as the only internal inhomogeneity, and the surface electrodes.

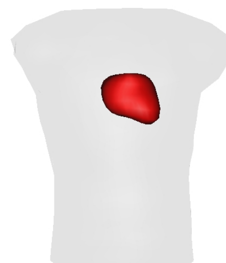

Fig. 4. Reconstructed geometry by the inverse solution.

3. RESULTS

We implemented our BEM approach using SCIRun [9] (a scientific programming environment) with a dynamic socket-based interface to MATLAB for some of the computations, and applied it to the model shown in Fig. 3 which is made of the heart and tank geometry in the SCIRun dataset. 192 electrodes distributed all over the torso surface are used in this model. In principle, it is possible to inject 191 independent current patterns and read the resulting voltages to be used for solving the inverse problem but in our simulation, to save time, as suggested in [1], we only used a linearly independent set of 100 current patterns which were chosen to have maximal information content (in the sense of being the best for distinguishing the medium from a homogeneous case). Specifically, we approximate the eigenfunctions with the first 100 largest eigenvalues of the difference between **T** for the model in Fig. 3 and that for a homogeneous model. In practice if we used fewer electrodes or had more unknowns, we could use the full number of independent current patterns to obtain all possible information to help the inverse solution.

To demonstrate the retrieval process as it would be applied to real data, we first generated mock noise-free torso data by applying the chosen current patterns to the electrodes and computing the resulting potentials on the same electrodes assuming $\sigma_1 = 1$ and $\sigma_2 = 10$ (arbitrary units) where σ_1 and σ_2 denote the conductivities of the torso and heart, respectively. We added spatially uncorrelated Gaussian noise, with a level like that expected in modern EIT system, *i.e.*, SNR=40dB. We used 50 control points to model the heart using cubic B-splines. The starting point for the iterative inverse solution was a homogeneous model, with a sphere in the center of torso as the initialization shape for the heart. Fig. 4 shows the reconstructed geometry and table 1 shows the percent

Table 1. Percent Error in Retrieved Conductivities

SNR=40dB , 5 Realizations	
$\Delta\sigma_1(\%)$	$\Delta\sigma_2(\%)$
1.42 ± 3.72	1.51 ± 2.87

error in retrieved conductivities for the error cost function (2). We used (3) as the regularization term in (2) and ran the simulation 5 times with different noise realizations.

4. DISCUSSION

In our formulation of the inverse problem of EIT when we assume a piece-wise constant conductivity profile and parameterize the surfaces of the internal inhomogeneities, the inverse problem becomes much better posed comparing to use the classical approach of voxelating the region of interest and treating the values of the conductivity in each voxel as an unknown. Using BEM, only the surfaces of objects need to be discretized, in contrast to discretizing the whole volume in, for example, Finite Element Method (FEM), often used in classical approaches. It is one of the major pluses of BEM, especially because construction of volume meshes for complicated objects, particularly in 3D , is a time consuming exercise. Another advantage of BEM is that only the boundary conditions are being approximated whereas in FEM the whole differential equation is being approximated [10]. A disadvantage of BEM is that it cannot easily model anisotropic conductivity regions, which might be a factor in the heart and skeletal muscle regions. Another disadvantage is that its integrals are far harder to evaluate than the element integrals in FEM. Also the integrals that are the most difficult (those containing singular integrands) have a significant effect on the accuracy of the solution. The need to evaluate integrals involving singular integrands makes the BEM at least an order of magnitude more difficult to implement than a corresponding FEM procedure. This, plus the fact that our inverse solution is a nested iterative algorithm, makes the reconstruction computationally expensive and time consuming. Using denser meshes for the surfaces increases the accuracy of the reconstruction with the price of increasing the solution time. In our simulation, we used 132 nodes and 260 triangles for the heart and 771 nodes and 1538 triangles for the torso surface, or in total 903 nodes and 1798 triangles to model the whole geometry. We designed our BEM code in a way that when we solve the forward problem for the first time, we keep those parts which are the same for all iterations in the inverse solution, so that we would not need to recompute them at each iteration. We ran the algorithm on a PC with an Intel(R) Pentium(R) 4 2.60GHz CPU, 1GB RAM and Red Hat Linux 3.2.2-5 as the operating system. The inverse solution took almost 10 hours to complete. The running time, of course, depends on the code; in this work even though we tried to do efficient programming, we did not specifically focus on optimizing the speed. In this algorithm the most computationally expensive part is calculating the elements of a number of the matrices [4], including the coefficient matrices and some of their derivatives. However the required matrices can all be computed independently of each other, and thus should be amenable to speed-up through parallelization.

Modeling the heart surface using basis shape functions with a small number of control points makes the reconstructed heart surface smoother than a real heart. We can get a better approximation of the heart surface by using more control points at the cost of increasing the number of unknowns, which would make the convergence slower and decrease the reconstruction accuracy. Like all other problems which use regularization techniques, finding the right regularization and constraint condition is not an easy task. Improving the regularization method is a subject of on-going work.

Unlike some work in EIT that current is injected using a pair of electrodes and voltage is measured using *other* electrodes, we use the *adaptive method* proposed in [11] where current is injected through all electrodes simultaneously and the voltages are measured using the same electrodes with respect to a single grounded electrode. Work is currently underway to impose the more accurate electrode boundary conditions described in [12] in our 3D BEM model.

5. REFERENCES

[1] M. Cheney, D. Isaacson, and J. C. Newell, "Electrical impedance tomography," *SIAM Review*, vol. 41, no. 1, pp. 85–101, 1999.

[2] Misha E. Kilmer, Eric L. Miller, Alethea Barbaro, and David Boas, "Three-dimensional shape-based imaging of absorption perturbation for diffuse optical tomography," *Applied Optics*, vol. 42, no. 16, pp. 3129–3144, June 2003.

[3] Olli-Pekka Tossavainen, M. Vauhkonen, L. M. Heikkinen, and T. Savolainen, "Estimation shapes and free surfaces with electrical impedance tomography," *Measurement Science and Technology*, vol. 15, pp. 1402–1411, 2004.

[4] Saeed Babaeizadeh, Dana H. Brooks, and David Isaacson, "A 3-D boundary element solution to the forward problem of electrical impedance tomography," in *Proceedings of the 26th Annual International Conference of the IEEE EMBS*, San Francisco, CA, USA, September 2004, pp. 960–963.

[5] C. De Boor, "Efficient computer manipulation of tensor products," *ACM Trans. Math. Software*, vol. 5, pp. 173–182, 1979.

[6] T. F. Coleman and Y. Li, "An interior, trust region approach for nonlinear minimization subject to bounds," *SIAM Journal on Optimization*, vol. 6, pp. 418–445, 1996.

[7] J. A. Nelder and R. Mead, "A simplex method for function minimization," *Comput. J.*, vol. 7, pp. 308–313, 1965.

[8] A. L. Piccolomini and F. Zama, "The conjugate gradient regularization method in computed tomography problems," *Appl. Math. and Comp.*, vol. 102, pp. 87–99, 1999.

[9] "http://software.sci.utah.edu/," .

[10] Peter Hunter and Andrew Pullan, "FEM/BEM notes," Department of Engineering Science, The University of Auckland, New Zealand, 2002.

[11] D. G. Gisser, David Isaacson, and J. C. Newell, "Theory and performance of an adaptive current tomography system," *Clinical Physics and Physiological Measurements*, vol. 9, pp. 35–41, 1988.

[12] E. Somersalo, M. Cheney, and David Isaacson, "Existence and uniqueness for electrode models for electric current computed tomography," *Inverse Problems*, vol. 52, no. 4, 1992.