POINT SPREAD FUNCTION OPTIMIZATION FOR MRI RECONSTRUCTION [1]

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ABSTRACT

Magnetic Resonance Imaging (MRI) requires reconstruction of an image from non-uniformly sampled Fourier domain data. The point spread function (PSF) is strongly affected by the Fourier weights, also called density compensation function (DCF). We formulate the DCF selection as an optimization problem with linear matrix inequality (LMI) constraints. The maximum sidelobe level of the PSF is minimized while the mainlobe width is held constant. Our approach is in contrast to existing suboptimal approaches where a DCF is first found based on local sampling density, and then a secondary windowing function is applied to reduce sidelobes. Reconstructions of simulated data demonstrate that our approach produce more accurate and visually better looking images.

1. INTRODUCTION

MRI is a powerful and widely adopted imaging modality in biomedical applications. In MRI, the raw data can be modeled as irregularly spaced samples in the Fourier domain (k-space) of the image. The reconstruction problem is then to estimate the image from the k-space samples. The most common method for MRI image reconstruction is gridding [1]-[2]. In this method, the nonuniformly sampled data points are first resampled onto a Cartesian grid by convolving with an interpolation kernel; then FFT is performed on the regridded data to obtain the reconstruction image. Another well-known method is conjugate phase (CP), where the irregular k-space samples are directly used and multiplied with a spatially dependent conjugate exponential factor, and the results summed for every spatial point. In both gridding and CP, a DCF must be applied to compensate for the non-uniform k-space sampling density.

Many researchers have worked on the MRI reconstruction and attempted to get high quality reconstruction images. Most of the works are based on gridding algorithm. Jackson *et. al*. have studied various interpolating kernels for gridding algorithm and find the best one in terms of the relative amount of aliasing energy [2]. Meyer *et. al.* derived DCF for gridding in spiral imaging [3]. Hoge *et. al*. proposed DCF based on the Jacobian determinant for the transformation between Cartesian coordinates and the nonuniform coordinates [4]. Hossein Sedarat *et. al*. established the relationship between the gridding algorithm and the least squares method and found the optimal gridding parameters by minimizing the average error of matrices approximation [5]. At the same time, algorithms other than gridding such as Pseudoinverse Calculation [6], Gegenbauer Reconstruction [7] and URS/BURS [8] have been studied. But none of these methods considered optimizing the PSF directly, though PSF is the most important characteristic of an imaging system.

We note that the mathematically ideal DCF may not produce the most desirable PSF because it may suffer from high sidelobes. In this paper we propose to reformulate the DCF selection as an optimization problem with LMI constraints. Our goal is to design a DCF that explicitly address the mainlobe/sidelobe trade off, i.e. minimize sidelobe level for a given mainlobe width. The prior knowledge for the region of interest (ROI) is also taken into consideration. If more prior knowledge is available and specific PSF shape is desirable, the proposed method can approximate the desired PSF in the minimax sense. Thus our method is a novel approach that provides flexible trade-off between contrast and resolution in the reconstructed image.

In section 2, we review the gridding and CP methods in MRI reconstruction, demonstrated the relationship between DCF and the PSF, and formulated the DCF design problem in the LMI framework. In section 3, we compare reconstruction results using the optimized DCF with those using prevailing DCFs. The advantages of our method will be demonstrated. In the final section, we summarize advantages of this work.

2. PROBLEM FORMULATION

2.1 MRI Image Reconstruction Methods

In MRI, the raw data can be modeled as nonuniform samples in the spatial frequency domain, or better known in the MRI community as k-space [9]. Figure 1 shows two trajectories widely used in non-Cartesian sampling: radial sampling and spiral sampling. The acquired data can be expressed as sampled 2D Fourier transform of the image:

$$F_n = F(k_{x,n}, k_{y,n}) = \int_{FOV} f(x,y)\exp\left(-j2\pi(xk_{x,n} + yk_{y,n})\right)dxdy \quad (1)$$

where x and y denote the position in image domain,

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$(k_{x,n}, k_{y,n}), n = 1,2,\ldots$ are sampling points in the k space, FOV is the field of view. The objective of reconstruction is to find a good estimate of the image $I(x,y)$ given the k-space data F_n.

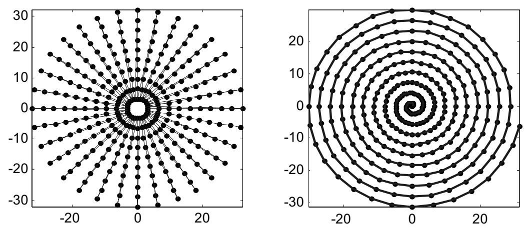


Fig.1: Radial (left) and spiral (right) k-space sampling trajectory

In the gridding method, nonuniformly measured data are first interpolated onto a uniform (rectangular) grid, then inverse Fast Fourier transform (FFT) is applied to the interpolated data. This procedure can be expressed as

$$\tilde{F}(k_x,k_y) = \sum_n F_n D(k_{x,n},k_{y,n}) H(k_x - k_{x,n}, k_y - k_{y,n}) \qquad (2)$$

where $H(k_x,k_y)$ is the interpolation kernel, $D(k_x,k_y)$ is the DCF inversely proportional to the local sampling density [5]. A widely used kernel is the Kaiser-Bessel window with $\beta = 13.9068$, proposed by Jackson [2]. Taking inverse Fourier transform of (2), we obtain:

$$\tilde{f}(x,y) = h(x,y) \sum_n F_n D(k_{x,n},k_{y,n}) \exp\left(j2\pi(xk_{x,n} + yk_{y,n})\right) \qquad (3)$$

where $h(x,y)$ is the inverse Fourier transform of $H(k_x,k_y)$. For a good interpolation kernel, it is generally true that $h(x,y)$ is strictly positive within the FOV. Finally, the reconstruction image is obtained by

$$\hat{f}(x,y) = \tilde{f}(x,y)/h(x,y) \qquad (4)$$

In the conjugate phase method, the k-space data points are weighted by the DCF and multiplied by a spatial dependent conjugate phase factor, then the results are coherently summed:

$$\hat{f}(x,y) = \sum_n F_n D(k_{x,n},k_{y,n}) \exp(j2\pi(xk_{x,n} + yk_{y,n})) \qquad (5)$$

Compared to the gridding method, the CP method is more accurate but slower, because FFT cannot be used. In both methods, it is easy to show that the reconstructed image can be expressed as the ground truth image convolved with a PSF,

$$\hat{f}(x,y) = f(x,y) * g(x,y) \qquad (6)$$

where the PSF is related to the DCF by

$$g(x,y) = \sum_n D(k_{x,n},k_{y,n}) \exp(j2\pi(xk_{x,n} + yk_{y,n})) \qquad (7)$$

The desired properties of the PSF are narrow mainlobe width and low sidelobe level. The mathematically ideal DCF given by Meyer [3] or Hoge [4] do not necessarily give good PSF because truncation and discretization in the k-space result in high sidelobes and grating lobes. An empirical window function can be applied to the k-space samples on top of the ideal DCF. However, such approach is neither rigorous nor optimal. Our approach is to formulate the DCF selection as a convex optimization problem with LMI. The DCF so designed can guarantee certain mainlobe width while minimizing the sidelobe level. For convenience, we write $w_n = D(k_{x,n},k_{y,n})$ and $V_n(x,y) = \exp(j2\pi(xk_{x,n} + yk_{y,n}))$. Then we want to design the weights w_n for the non-uniform k-space points $(k_{x,n},k_{y,n})$ so that the PSF expressed in (7) has the desired properties. The problem presented here is similar to the beam pattern design problem for non-uniform arrays [10], although the details and objective differ. Below we give a brief derivation of the LMI design formulation.

2.2 Designing DCF with LMI

A Linear Matrix Inequality is any constraint of the form

$$A(\mathbf{x}) := A_0 + x_1 A_1 + \ldots + x_N A_N > 0 \qquad (8)$$

where

- $\mathbf{x} = (x_1, \ldots, x_N)$ is a vector of unknown scalars (the decision or optimization variables)
- $A_0, \ldots, A_N$ are given symmetric matrices
- >0 stands for "positive definite," i.e., the smallest eigenvalue of $A(\mathbf{x})$ is positive [11], [12]. LMI constraints are convex, thus many optimization problems can be solved numerically by efficient convex optimization algorithms [10-12].

Suppose we wish to minimize the maximum sidelobe level in the FOV given a mainlobe width Δ. This corresponds to maximizing image contrast for a given resolution. We may express this objective mathematically as

$$\min_{\mathbf{w}} \ \varepsilon, \quad \text{subject to} \begin{cases} g(0,0) \equiv \sum_n w_n = 1 \\ \left| \sum_n w_n V_n(x,y) \right|^2 \le \varepsilon \quad \text{for} \quad \Delta \le \sqrt{x^2 + y^2} \le L \end{cases} \qquad (9)$$

where L is the radius of the FOV. The second constraint in Eq. (9) can further be expressed as a LMI

$$\begin{bmatrix} \varepsilon & \mathbf{w}^{\mathrm{H}} \mathbf{V}(x,y) \\ \mathbf{V}(x,y)^{\mathrm{H}} \mathbf{w} & 1 \end{bmatrix} \ge 0 \qquad (10)$$

where $\mathbf{w} = [w_1, w_2, \cdots]^{\mathrm{T}}$ is the DCF vector to be optimized and $\mathbf{V}(x,y) = [V_1(x,y), V_2(x,y), \cdots]^{\mathrm{T}}$. Direct application of the LMI constraint (10) is computationally expensive for a 2D problem. For example, consider reconstruction of a 100x100 image from 3000 k-space samples. This would incur 3000 design variables with 10000 constraints. We may mitigate the computational burden by using approximate circular symmetries of the k-space samples and the PSF. First, the PSF is approximately circularly symmetric, therefore only constraints along a radial slice need to be included. Second, the k-space samples are approximately circularly symmetric, therefore we may assign only one weight for each radius. For example, in spiral sampling, the sampling points of the trajectory can be expressed as

$$\begin{aligned} k_{x,n} &= (nk_{\max} / N_k)\cos(2n\pi / N_t) \\ k_{y,n} &= (nk_{\max} / N_k)\sin(2n\pi / N_t) \end{aligned} \tag{11}$$

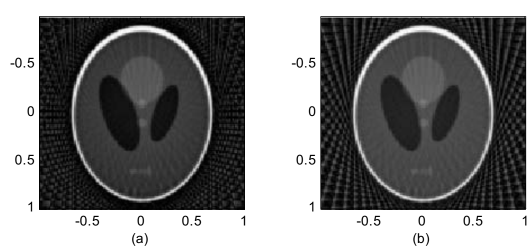
We may assign w_n to be the same for $(m-1)N_t \le n < mN_t$. We name these weights w_m and then define

$$V_m(x,y) = \sum_{n=(m-1)N_t}^{mN_t-1} \exp(j2\pi(xk_{x,n} + yk_{y,n})) \tag{12}$$

These new design variables and basis functions can then be substituted into (10) and solved effectively with existing techniques.

3. SIMULATION EXAMPLES

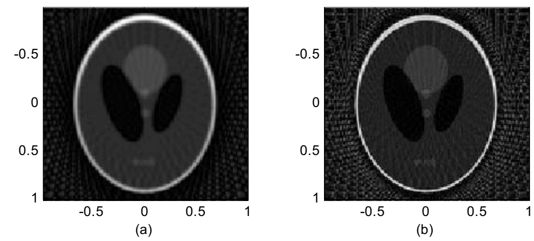
In this section we present reconstruction examples from synthetic data for the modified Shepp-Logan phantom [13]. The phantom consists of ellipses of different contrast, and its Fourier transform can be analytically calculated [6] and sampled at arbitrary points. We use the Matlab LMI toolbox as the optimization solver. To measure the reconstruction accuracy, we calculated the correlation coefficient ρ of the reconstruction image and the phantom image. An equivalent measure would be the normalized mean square error (NMSE), which is related to the correlation coefficient by $\text{NMSE} = 1 - \rho^2$. We have considered both radial and spiral sampling, with $N_k = 6400$ sampling points filling a bandwidth of $k_{\max} = 32$ in each case. As a benchmark for comparisons, reconstruction images using traditional gridding method and the DCF proposed by Jackson *et. al* [2] are shown in figure 2, for radial and spiral samplings respectively. The artifacts due to PSF sidelobes are obvious here.



(a) Radial sampling $\rho = 0.814$ (b) Spiral sampling $\rho = 0.803$

Fig. 2: Gridding reconstruction using Jackson's DCF

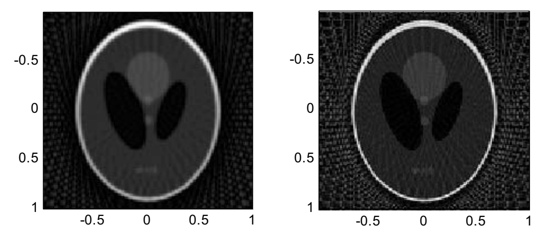
Next we design the optimal DCF using the LMI technique discussed in section 2. For radial sampling, we choose $\Delta = 1.2 / k_{\max}$ and $L = 0.36$ in Eq. (9). The minimum sidelobe level achieved with this design was $\varepsilon = \text{-58.3dB}$. Using the LMI optimized DCF, we reconstruct the phantom with both conjugate phase and gridding methods respectively. The images are shown in figures 3-4, along with their respective correlation coefficients. For comparison, the same reconstructions using Hoge's DCF [4], which is mathematically exact, are shown side by side.



(a) LMI optimized $\rho = 0.900$ (b) Hoge's DCF $\rho = 0.800$

Fig. 3: CP Reconstruction for radial sampling

We observe that with both methods the reconstruction image using our optimized DCF has better visual appearance as well as higher correlation coefficient than that using Hoge's DCF, with improvements more dramatic in the conjugate phase method. Figure 5, which is a cross sectional plot of the phantom and CP reconstructions, shows the advantages of the optimized DCF more clearly.



(a) LMI optimized $\rho = 0.825$ (b) Hoge's DCF $\rho = 0.670$

Fig. 4: Regridding reconstruction for radial sampling

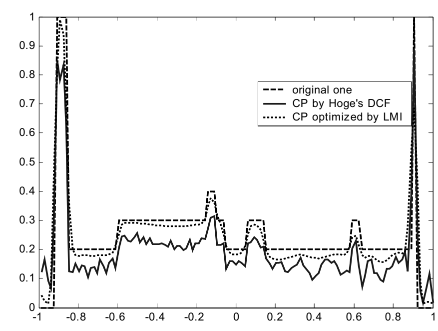
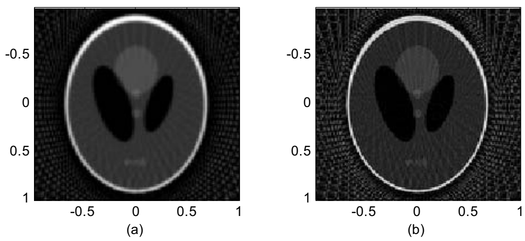


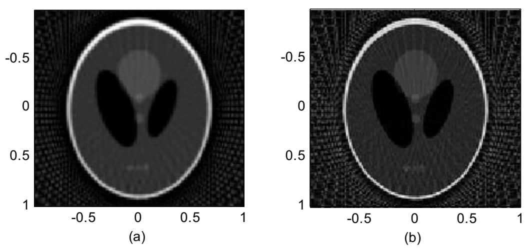
Fig. 5: Centerline profile comparison for radial sampling

For spiral sampling, we choose $\Delta = 1.1 / k_{\max}$ and $L = 0.30$. The minimum sidelobe level achieved was $\varepsilon = -55.4\text{dB}$. We perform reconstructions using the LMI optimized DCF with gridding and conjugate phase methods respectively. The results are shown in figures 6-7. Also for comparison, reconstructions using Meyer's DCF [3], which is mathematically exact, are shown side by side. Similar to the case of radial sampling, the optimized DCF results in higher quality reconstruction both visually and quantitatively.



(a) LMI optimized $\rho = 0.911$ (b) Meyer's DCF $\rho = 0.826$

Fig. 6 CP Reconstruction results for spiral sampling



(a) LMI optimized $\rho = 0.828$ (b) Meyer's DCF $\rho = 0.702$

Fig. 7 Regridding results for spiral sampling

It should be noted that a sidelobe-reducing window (e.g. Hamming window) could be applied on top of Hoge's or Meyer's DCF. However, such approach is empirical and not optimized. For example, in spiral sampling, applying the Hamming window on top of Meyer's DCF results in wider mainlobe and lower sidelobes. However adjusting our optimization parameters, we could obtain a PSF with the same mainlobe width, but lower sidelobe level, in the same ROI. This effect is clearly demonstrated in figure 8. The main strength of our method is that it allows flexible yet optimum tradeoff between mainlobe width and sidelobe level, or between contrast and resolution. Moreover, it is applicable to sampling trajectories where no analytical DCF can be derived.

4. SUMMARY AND CONCLUSIONS

We have formulated the MRI image reconstruction problem as a PSF optimization problem with LMI constraints. Compared to the traditional approach of obtaining an accurate DCF and subsequently applying a window, our method directly optimizes individual weights of the k-space sampling points with the objective expressed as obtaining the minimum sidelobe level for a given mainlobe width of the PSF, or equivalently obtaining the minimum mainlobe width for a given sidelobe level. The most important advantage of the proposed method is its flexibility to control the tradeoff between spatial resolution and contrast resolution. We demonstrated that the reconstructed images using the optimized DCF are visually better and numerically more accurate than those obtained with existing methods.

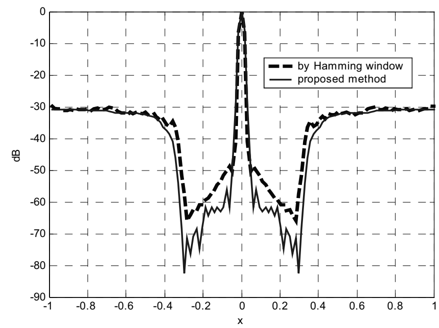


Fig. 8: Comparison of optimized PSF (solid) and that of Meyer's DCF plus Hamming window (dashed)

5. REFERENCES

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