SIGNAL-DEPENDENT ERROR IN SPECULAR MULTIPATH ESTIMATION

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ABSTRACT

Signal-dependent error for least squares specular multipath estimators is analyzed for active sensors. The error analysis also applies to other deconvolution techniques for multipath estimation such as Projection Onto Convex Sets (POCS) that use square error as a constraint. The results explain how the selection of transmit signal affects estimator performance. Example mean square error plots are provided for a linear frequency modulation (LFM) signal to demonstrate the results. Given the set of possible locations for multipath components, the optimum signal to minimize square error can be selected using these results.

1. INTRODUCTION

In active sonar and radar systems, estimation of the multipath response is the critical element of a variety of algorithms for enhanced sensor performance. Time variation in the sensor environment requires algorithms that estimate both the delay and Doppler of multipath components. The Doppler estimate is also useful for distinguishing direct from reflected paths. This paper considers a non-stationary environment that requires a rapid estimate of the multipath response using a single transmitted pulse.

For active sensors, the selection of the transmit signal determines the resolution performance of the system. The *narrowband ambiguity function*, defined as

$$\chi(\tau,\upsilon) = \int_{-\infty}^{\infty} x(t)x^*(t+\tau)e^{-2\pi j\upsilon t}\,dt \qquad (1)$$

is a measure of sensor resolution for the case with two specular targets separated in range delay by τ and Doppler frequency by υ [1]. For multiple specular targets/reflectors, analysis of sensor resolution requires a more complex treatment to address all of the interactions between components.

In this paper, mean square estimation error is analyzed as a measure of sensor resolution performance. Least squares deconvolution is used for the estimator. This technique has been used for specular multipath estimation in a variety of applications [2-4]. The least squares error analysis is also relevant to other iterative techniques such as Projection Onto Convex Sets (POCS) that use square error as a constraint [5-8]. The error analysis is important for selection of the transmit signal based on known *a priori* characteristics of the channel, and it is important for analyzing error performance of deconvolution-based estimators.

This paper is organized as follows: Section 2 describes the channel model. Section 3 reviews the least squares delay-Doppler estimator. The error analysis is described in Section 4. Section 5 presents results of applying the error analysis to a linear frequency modulation (LFM) transmit signal. Section 6 concludes the paper with a summary of results.

2. CHANNEL MODEL

With N specular reflectors and a narrowband signal, active sensor operation within a time-varying multipath environment is modeled as

$$y(t) = \sum_{k=1}^{N} \alpha_k e^{-j2\pi\nu_k t} x(t-\tau_k) + w(t) \qquad (2)$$

where y(t) is the receive signal, x(t) is the transmit signal, w(t) is white noise, and the triplet $(\alpha_k, \nu_k, \tau_k)$ describes the complex amplitude, narrowband Doppler shift, and delay of the kth specular multipath component. This is a deterministic spreading function model of the channel. Time variation of $(\alpha_k, \nu_k, \tau_k)$ is negligible given an adequately short observation interval.

A discrete-time version of (2) is

$$\mathbf{y} = \mathbf{Ah} + \mathbf{w} \qquad (3)$$

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where **y** is the received signal, **A** is a signal matrix, **h** is the channel response to be estimated, and **w** is white noise with

$$E[\mathbf{w}\mathbf{w}^H] = \sigma^2 \mathbf{I} \quad .$$

Each column of **A** contains a time- and frequency-shifted version of the transmit signal representing the received signal from a different specular multipath component. This matrix can be generalized to model wideband conditions by modifying the columns to hold delayed and time-scaled versions of the transmit signal.

The matrix is formed using the method described in [3]. A Doppler sub-matrix is created,

$$\mathbf{A}_0 = \mathbf{s}(\tau_0, \upsilon_0)..\mathbf{s}(\tau_0, \upsilon_{n_v-1}) \ ,$$

where each column contains one of n_v different Doppler shifts of the undelayed transmit signal. This matrix is then extended by applying the n_d multipath delays by zero padding and shifting the A_0 matrix:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{0}^T & \dots & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{A}_0 & \dots & \mathbf{0}^T \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{0}^T & \mathbf{0}^T & \dots & \mathbf{A}_0 \end{bmatrix} . \qquad (4)$$

The columns of **A** contain the received signals from all potential unit-amplitude multipath components. The **h** vector is an $n_d n_v$ x 1 column vector containing the amplitude of each multipath component:

$$h_d = \begin{bmatrix} h(d,0) \\ h(d,1) \\ \vdots \\ h(d, n_v - 1) \end{bmatrix} \qquad \mathbf{h} = \begin{bmatrix} \mathbf{h}_0 \\ \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_{n_d-1} \end{bmatrix} . \qquad (5)$$

The model represents the observed signal as a mixture of the component receive signals with additive white noise, i.e. the **h** vector expresses a linear combination of the columns of **A** that is combined with a white noise vector **w** to form the observed signal **y**.

3. LEAST SQUARES DECONVOLUTION

Least squares algorithms for multipath deconvolution produce the multipath estimate that minimizes the norm residual error, providing the estimate that "best explains" the received data in a square error sense. This procedure has been used for delay-only multipath estimation in [2] and extended to delay and Doppler in [3].

To improve performance and reduce the effects of ill-conditioning, an indicator set is introduced to limit support to regions of strong matched-filter output. The estimate is set to zero for regions with matched filter output magnitude below a threshold. The indicator set support constraint can be applied with the matrix multiplication

$$A_I = A I_E \ ,$$

where each column of I_E has exactly one element equal to one and all other elements equal to zero. The position of the unity element determines the column selected from A. The effect is to eliminate multipath components from the model. With the simplified model, the least squares estimate is then obtained as

$$\hat{\mathbf{h}}_I = \left(\mathbf{A}_I{}^H \mathbf{A}_I\right)^{-1} \mathbf{A}_I{}^H \mathbf{y} \, . \qquad (6)$$

Equation (6) is a pseudo-inverse applied with the simplified signal matrix, A_I. In white noise this yields the minimum mean square error estimate.

4. ERROR ANALYSIS

The square-error of least-squares deconvolution can be derived using the singular value decomposition. The mean square error (MSE) is

$$MSE = E[(\hat{\mathbf{h}} - \mathbf{h})^H (\hat{\mathbf{h}} - \mathbf{h})] \, .$$

Substituting (3) yields

$$MSE = E[\mathbf{w}^H \mathbf{A}_I \left(\mathbf{A}_I^H \mathbf{A}_I\right)^{-2} \mathbf{A}_I^H \mathbf{w}] \qquad (7)$$

After substituting the singular value decomposition,

$$\mathbf{A}_I = \mathbf{U}\mathbf{S}\mathbf{V}^H \, , \qquad (8)$$

into (7), where **U** and **V** are orthonormal and **S** is a rectangular matrix of real singular values described by

$$S_{ij} = \begin{cases} \lambda_i, & i = j \\ 0, & i \neq j \end{cases} ,$$

the MSE is

$$MSE = \sum_{i=0}^{N-1} \frac{\sigma^2}{\lambda_i^2} \ , \qquad (9)$$

where N is the size of the indicator set. Since the signal matrix, **A**, has normalized columns, λ_i must be less than or equal to one. Consequently the minimum MSE is $N\sigma^2$.

The mean square error in (9) assumes the indicator set is known *a priori*. In a real system, this information is not known since this depends on the distribution and amplitude of multipath components in the environment. However, given approximate characteristics of the multipath environment, a typical indicator set can be analyzed. For example the indicator set for a cluster of tightly grouped multipath components can be estimated using the magnitude of the ambiguity function for the signal [4].

Another approach to the error analysis is to transform the signal matrix such that it is an orthogonal matrix. In the transformed domain the multipath components are orthogonal and do not interact with each other. The reduction in norm of each transformed column represents a decrease in effective transmitted signal energy for that multipath component and leads to an increase in mean square error. The set of column norms is the set of λ_i in (9).

To derive the transformation, consider the matrix $A^H A$ as the N component discrete ambiguity function. Each column of $A^H A$ is a vector of inner products of the multipath component for that column with all of the multipath components. The matrix $A^H A$ can be diagonalized using an orthonormal matrix V and expressed in the form[1]

$$(\mathbf{AV})^H \mathbf{AV} = \mathbf{S} \ ,$$

where **S** is diagonal. Equation (3) can then be rewritten as

$$\mathbf{y} = (\mathbf{AV})(\mathbf{V}^H \mathbf{h}) + \mathbf{w}$$

After substituting

$$\mathbf{A}_t = \mathbf{AV}$$
$$\mathbf{h}_t = \mathbf{V}^H \mathbf{h}$$

a transformed version of (3) is obtained:

$$\mathbf{y} = \mathbf{A}_t \mathbf{h}_t + \mathbf{w} \ . \qquad (10)$$

[1] The SVD will yield an orthonormal V that produces a real-valued diagonalized S. Other transformations will also work if S is allowed to be complex.

The norms of the columns of $\mathbf{A}_t$ provide the set of λ_i to determine mean square error from (9). Minimum mean square error ($N\sigma^2$) is obtained when **A** starts as an orthonormal matrix.

5. RESULTS WITH LFM

The mean square error for least squares deconvolution was analyzed for an LFM transmit signal using (9). Two different specular multipath grids were considered. LFM bandwidth and pulse width were varied. The first case used the grid shown in Figure 1.

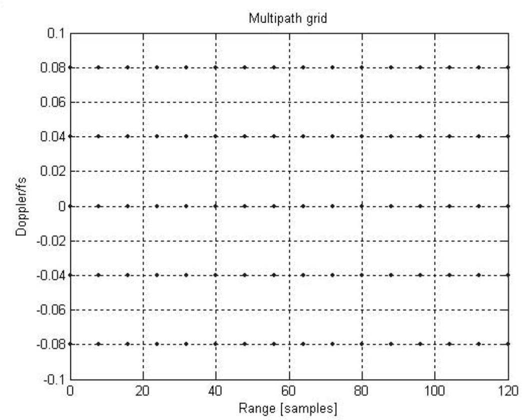

Figure 1. Uniform specular multipath grid

The grid in Figure 1 has 80 multipath scatterers uniformly spaced by eight samples in range and 0.04 times the sampling frequency in Doppler. Mean square error is plotted as a function of LFM bandwidth in Figure 2 and LFM pulse width in Figure 3.

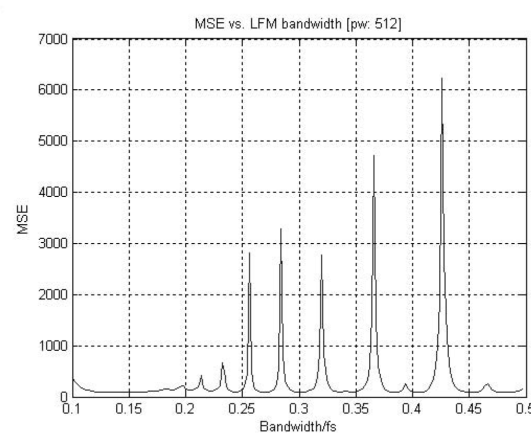

Figure 2. Mean square error vs. LFM bandwidth for uniform specular multipath grid. Minimum error is 83.7.

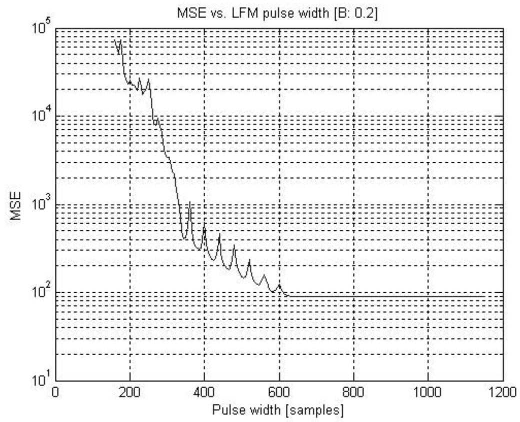

Figure 3. Mean square error vs. LFM pulse width for uniform specular multipath grid.

The peaks in Figures 2 and 3 correspond to cases where the ridge of the ambiguity function matches the slope of a sub-lattice within the grid. These are cases where the received signals from different scatterers are nearly identical (ambiguity function magnitude close to one), causing the signal matrix to become ill-conditioned and the MSE to grow.

The ill-conditioning effect is more pronounced with the sloped grid of Figure 4. The results for this grid are shown in Figure 5.

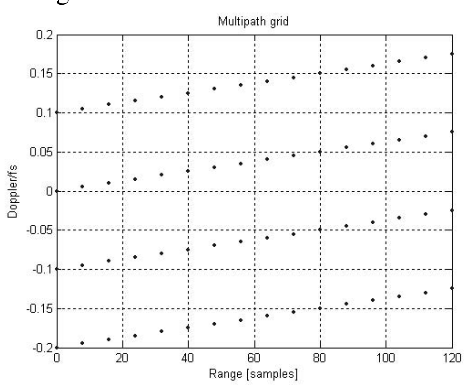

Figure 4. Grid with linear range-Doppler variation.

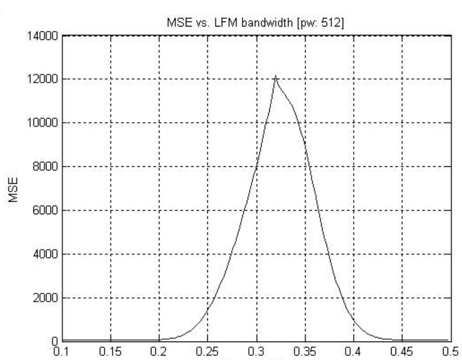

Figure 5. Mean square error vs. LFM bandwidth for multipath grid with linear range-Doppler variation.

In Figure 5 the broad error peak occupies the bandwidth region where the slope of the ambiguity ridge is close to the Doppler-range slope within the grid.

6. CONCLUSION

The error for a specular multipath estimator using least squares deconvolution was derived using the singular value decomposition. The error analysis also applies to other deconvolution techniques for multipath estimation such as POCS that use square error as a constraint. The mean square error for least squares deconvolution was analyzed for an LFM transmit signal. Plots of MSE vs. LFM with varying bandwidth and pulse width were presented. The minimum error for these cases was within 5% of the lower bound on MSE given by the error analysis.

7. REFERENCES

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