REALITY PRESERVING FRACTIONAL TRANSFORMS

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ABSTRACT

The unitarity property of transforms is useful in many applications (source compression, transmission, watermarking, to name a few). In many cases, when a transform is applied on real-valued data, it is very useful to obtain real-valued coefficients (i.e. a reality-preserving transform). In most applications, the decorrelation property of the transform is of importance and it would be very useful to control it under some transform's parameter (e.g. in joint source-channel coding).
This paper focuses on fractional transforms, as tools for obtaining such properties. We propose a methodology for obtaining them and obtain variants of the Discrete Fractional Cosine (Sine) Transform which share real-valuedness as well as most of the properties required for a fractional transform matrix. As shown in [17], such matrices cannot be symmetric.

1. INTRODUCTION

Several unitary transforms widely used in Signal Processing received fractional formulations: the Fourier transform, e.g. in [1, 2, 3], and other unitary transforms, e.g. Cosine and Sine [4, 5], Hartley [1, 5, 6], Hadamard [7], Hilbert [8, 9] and Legendre [10]. Basic properties required for a fractional matrix $\mathbf{A}_a$, with a the fractional real parameter varying in $[0,1]$, from here on *Basic Properties*, are: *orthogonality*; the *base matrix* $\mathbf{A}$ obtainable as $\mathbf{A}_1$; the *identity matrix* $\mathbf{I}$ obtainable as $\mathbf{A}_0$; *reality preserving* : if x is real-valued, so does the transformed version X of x (as e.g. for the Cosine and the Hartley Transforms); *parameter continuity*, i.e. with respect to a; *parameter additivity*: $\mathbf{A}_a\mathbf{A}_b = \mathbf{A}_{a+b}$; *parameter commutativity*: $\mathbf{A}_a\mathbf{A}_b = \mathbf{A}_b\mathbf{A}_a$; *parameter periodicity*, the period being defined as the minimum positive natural number p such that $\mathbf{A}_{a+p} = \mathbf{A}_a$: thus the fractional transform is defined and continuous also for a outside the interval $[0,1]$.
No known fractional transform matrix possesses all of them. In Section 2 we shortly recall the DCT-I , the DST-I as well as known results on Discrete Fractional Fourier, Cosine and Sine transforms. In Section 3 we present two methods to obtain fractional transform matrices possessing most of the *Basic properties*. We show experimentally that the resulting transforms can have continuously increasing decorrelation power when a varies from 0 (the identity, no decorrelation) to 1 (the base transform, which can be almost maximally decorrelating).

2. RELATED WORK

Two well-known reality preserving transforms are the DCT and the DST [11]. The DCT-I and the DST-I have kernel that can be written as

$$\sqrt{\frac{2}{N-1}}k_m k_n cos(\frac{mn\pi}{N-1}),\ \sqrt{\frac{2}{N}}sin(\frac{mn\pi}{N})$$

respectively, with $m,n = 0,1,\ldots,N-1$, $k_m = \frac{1}{\sqrt{2}}$ for $m = 0, N-1$, $k_m = 1$ otherwise for the DCT-I, while $m,n = 1,\ldots,N$ for the DST-I. We name such matrices $\mathbf{C}$ and $\mathbf{S}$ respectively and use $\mathbf{T}$ for either of them. In the eigenfactorization $\mathbf{T} = \mathbf{V}_T\mathbf{\Lambda}_T\mathbf{V}_T^t$, where $.^t$ denotes matrix transposition, $\mathbf{V}_T = [\mathbf{v}_0|\mathbf{v}_1|\ldots|\mathbf{v}_{N-1}]$ is a real matrix composed of orthogonal eigenvectors of $\mathbf{T}$. The diagonal $\mathbf{\Lambda}_T$ contains the eigenvalues of $\mathbf{T}$, i.e. 1 and -1. If N is even, the eigenvalue's multiplicity is $\frac{N}{2}$ for both 1 and -1. If N is odd, it is $\frac{N+1}{2}$ for 1 and $\frac{N-1}{2}$ for -1 ([4]). In [4], the method used by the same authors in [1, 2] to obtain the Discrete Fractional Fourier Transform matrix $\mathbf{F}_a = \mathbf{V}_F\mathbf{\Lambda}_F^a\mathbf{V}_F^t$, with $\mathbf{\Lambda}_F^a$ having $e^{-ina\frac{\pi}{2}}$ as eigenvalues, is used to obtain $\mathbf{T}_a = \mathbf{V}_T\mathbf{\Lambda}_T^a\mathbf{V}_T^t$. Unique eigenvectors are obtained from the even Hermite-Gauss eigenvectors $\mathbf{V}_F$ of the Fourier matrix $\mathbf{F}$ in the cosine case, while from the odd ones in the sine case. $\mathbf{T}$ is obtained for $a = 1$, $\mathbf{I}$ for $a = 0$. In [1] the same method has been used to obtain a fractional, not reality preserving, Hartley transform. $\mathbf{T}_a$ enjoys all the *Basic Properties* except reality preserving for $a \neq \pm 1$ and $a \neq 0$ and moreover it is symmetric. It is 2-periodic (instead of 4-periodic as $\mathbf{T}$ is). In [5, 12, 13, 14] reality preserving fractional transforms have been defined. All these transforms do not have some of the remaining *Basic Properties*.

3. NEW FRACTIONAL TRANSFORMS

Here we first formulate a method (Method I) to obtain some reality preserving discrete fractional cosine and sine transforms and then a method to define a reality preserving fractional transform via a discrete fractional linear transform that complexifies in the transform domain. Following this method with the Fourier Transform we obtain a discrete fractional real Fourier transform which is the fractionalization of a "Real Fourier Transform". The computational complexity of both methods is $O(N^2)$, with N the matrix order.

3.1. Method I

A fractional matrix $\mathbf{T}_{a,N}$, $-1 \leq a \leq 1$, $N = 2^p, p \geq 2$, is obtained by the following main steps:
Step 1 - Construct $\mathbf{T}_N$, with $N = 2^p$, with $p = 2, 3, ...$
Step 2 - Construct $\mathbf{V}_{T,N}\mathbf{\Lambda}_{T,N}\mathbf{V}^t_{T,N} = \mathbf{T}_N$ with $\mathbf{\Lambda}_{T,N} = \mathbf{\Lambda}_{S,N} = \mathbf{\Lambda}_{C,N}$ ordered so that it has first the eigenvalue 1 until row $N/2$ and then -1 from row $N/2 + 1$ until row N.
Step 3 - Construct $\mathbf{V}_{T,N}(\mathbf{G1}_{N/2}(\theta(a)) \oplus \mathbf{G2}_{N/2}(\eta(a)))\mathbf{V}^t_{T,N} = \mathbf{T}_{a,N}$, with $\mathbf{G1}_{N/2}$, $\mathbf{G2}_{N/2}$ block-diagonal Givens matrices whose blocks are, respectively,

$$\mathbf{G1}_2(\theta(a)) = \begin{bmatrix} cos(\theta(a)) & sin(\theta(a)) \\ -sin(\theta(a)) & cos(\theta(a)) \end{bmatrix}$$

$$\mathbf{G2}_2(\eta(a)) = \begin{bmatrix} cos(\eta(a)) & sin(\eta(a)) \\ -sin(\eta(a)) & cos(\eta(a)) \end{bmatrix}$$

where $\theta(a) \neq \eta(a)$, a the fractional parameter, denote angles. $\mathbf{V}_{T,N} = [\mathbf{v}_0|\mathbf{v}_1|\ldots|\mathbf{v}_{N-1}]$ is composed of column eigenvectors of $\mathbf{T}$ and

$$\mathbf{G1}_{N/2}(\theta(a)) \oplus \mathbf{G2}_{N/2}(\eta(a))$$
$$= \begin{bmatrix} \mathbf{G1}_{N/2}(\theta(a)) & \mathbf{O} \\ \mathbf{O} & \mathbf{G2}_{N/2}(\eta(a)) \end{bmatrix}$$

fractionally generalizes the diagonal $\mathbf{\Lambda}_{T,N}$. $\mathbf{T}_{a,N}$ is not symmetric for $\theta(a) \neq \pm 2k\pi$ and $\eta(a) \neq \pm(2k+1)\pi$, $k = 0, 1, 2, \ldots$. The infinite set of matrices that has been defined is such that for $a = 1$, $\theta(1) = \pm 2k\pi$, $\eta(1) = \pm(2k+1)\pi$ and the fractional transform reduces to the original one $\mathbf{T}$. For $a = 0$, the fractional transform reduces to the identity matrix. To select one fractional matrix of that family as the intended unique one, a particular orthonormal complete set of vectors for $\mathbf{V}_{T,N}$ has to be fixed.

3.1.1. DFrCT matrices

We limit us to consider three DFrCT (Discrete Fractional Cosine Transforms) obtained from the DCT-I as base matrix $\mathbf{C}$, and sketch the verification of the *Basic Properties* they possess.
DFrCT.1 DFrCT.1 matrices $\mathbf{C}_{a,N}$ have

$$\mathbf{G1}_{N/2}(2a\pi) \text{ and } \mathbf{G2}_{N/2}(a\pi).$$

Basic Properties:
Orthogonality: $\mathbf{C}_{a,N}\mathbf{C}^t_{a,N} = \mathbf{I}_N$. *Base matrix*: $\mathbf{C}_{a,N} = \mathbf{C}_N$ for $a = 1$ since $\mathbf{C}_{1,N} = \mathbf{V}_{C,N}(\mathbf{I}_{N/2} \oplus -\mathbf{I}_{N/2})\mathbf{V}^t_{C,N} = \mathbf{C}_N$. *Identity matrix*: $\mathbf{C}_{a,N} = \mathbf{I}_N$ for $a = 0$ because $\mathbf{C}_{0,N} = \mathbf{V}_{C,N}(\mathbf{I}_{N/2} \oplus \mathbf{I}_{N/2})\mathbf{V}^t_{C,N} = \mathbf{I}_N$. *Reality preserving*, by construction. *Parameter continuity*, by definition. *Parameter additivity*: $\mathbf{C}_{a,N}\mathbf{C}_{b,N} = \mathbf{C}_{a+b,N}$ as it is easy to verify. *Parameter commutativity*: $\mathbf{C}_{a,N}\mathbf{C}_{b,N} = \mathbf{C}_{b,N}\mathbf{C}_{a,N}$, being $\mathbf{C}_{a,N}$ additive and $+$ commutative. *Parameter periodicity* with $p = 2$, i.e. $\mathbf{C}_{a+2,N} = \mathbf{C}_{a,N}$. In fact, $\mathbf{C}_{a+2,N} = \mathbf{V}_{C,N}(\mathbf{G1}_{N/2}(2(a+2)\pi) \oplus \mathbf{G2}_{N/2}((a+2)\pi))\mathbf{V}^t_{C,N} = \mathbf{V}_{C,N}(\mathbf{G1}_{N/2}(2a\pi + 4\pi) \oplus \mathbf{G2}_{N/2}(a\pi + 2\pi))\mathbf{V}^t_{C,N} = \mathbf{C}_{a,N}$.
DFrCT.2 DFrCT.2 matrices $\mathbf{C}_{a,N}$ have

$$\mathbf{G1}_{N/2}(\frac{1}{2}a(1-a)\pi) \text{ and } \mathbf{G2}_{N/2}(a\pi).$$

Basic properties: as for the DFrCT.1, except parameter additivity, that holds only for particular values of a. A *quasi-additivity* property holds as defined in [17].
DFrCT.3 DFrCT.3 matrices $\mathbf{C}_{a,N}$ have

$$\mathbf{G1}_{N/2}(\frac{1}{2}a(1-a)\pi) \text{ and } \mathbf{G2}_{N/2}(\frac{1}{2}a(1+a)\pi).$$

Basic properties: as for the DFrCT.2 case.

3.1.2. Other fractional cosine matrices

We obtained by Method I other fractional cosine matrices based on the DCT-IV (a shifted version of the DCT-I) with kernel

$$\sqrt{\frac{2}{N}}cos(\frac{(m+1/2)(n+1/2)\pi}{N})$$

$m, n = 0, 1, \ldots, N-1$. Such matrices correspond to the DFrCT.1, DFrCT.2 and DFrCT.3 and have quite similar residue correlation curves, as expected. The DCT-II matrix (as its transpose DCT-III matrix) has not been considered because Method I requires a symmetric base matrix to obtain an eigendecomposition of the form $\mathbf{V\Lambda V}^t$. Method I allows to construct also fractional transform matrices based on the DCT-II one, with expectedly analogous decorrelation power, as long as we accept to miss some of the *Basic Properties*, namely parameter additivity.

3.1.3. Fractional sine matrices

Three fractional sine transforms DFrST.1, DFrST.2 and DFrST.3 can be analogously obtained from the DST-I. Their $\mathbf{S}_{a,N}$ matrices are similar to those of the respective cosine cases. They are obtainable by replacing $\mathbf{V}_{C,N}$ with the eigenvector matrix $\mathbf{V}_{S,N}$ in the respective cosine definitions. The *Basic Properties* they share are similar as well. Sine fractional matrices based on the DST-IV can be similarly obtained.

3.2. Method II

Let $\mathbf{M}_{a,N/2}$ be a complex-valued fractional transform matrix with size $N/2$, N even. Let, in the following diagram

$$\mathbf{x} --> \mathbf{x}' --> \hat{\mathbf{x}} \longrightarrow \hat{\mathbf{y}} --> \mathbf{y}' --> \mathbf{y}$$

$\mathbf{x} = \{x_0, x_1, \ldots, x_{N-2}, x_{N-1}\}^t$ be a real signal,
$\mathbf{x}' = \{x'_0, x'_1, \ldots, x'_{N-2}, x'_{N-1}\}^t$ be a permutation of $\mathbf{x}$, that is $\mathbf{x}' = \mathbf{Px}$, with $\mathbf{P}$ a permutation matrix;
$\hat{\mathbf{x}} = \{x'_0 + ix'_{N/2+1}, \ldots, x'_{N/2-1} + ix'_{N-1}\}^t$, with length $N/2$, N even, be the complex vector built from $\mathbf{x}$ as it is here shown, $\hat{\mathbf{y}} = \mathbf{M}_a\hat{\mathbf{x}}$, $\mathbf{y}' = (Re(\hat{\mathbf{y}}), Im(\hat{\mathbf{y}}))$ and $\mathbf{y} = \mathbf{P}^{-1}\mathbf{y}'^t$. Thus,

$$\mathbf{y} = \mathbf{R}_a\mathbf{x} = \mathbf{P}^{-1}\mathbf{B}_a\mathbf{Px}$$

$$\mathbf{B}_a = \begin{bmatrix} Re(\mathbf{M}_a) & -Im(\mathbf{M}_a) \\ Im(\mathbf{M}_a) & Re(\mathbf{M}_a) \end{bmatrix}$$

is obtained from

$$\mathbf{M}_{a,N/2} + i\mathbf{M}_{a,N/2}$$
$$= Re(\mathbf{M}_{a,N/2}) + iIm(\mathbf{M}_{a,N/2}) + i(Re(\mathbf{M}_{a,N/2}) + iIm(\mathbf{M}_{a,N/2}))$$
$$= Re(\mathbf{M}_{a,N/2}) - Im(\mathbf{M}_{a,N/2}) + i(Re(\mathbf{M}_{a,N/2}) + Im(\mathbf{M}_{a,N/2}))$$

with $\mathbf{Px}$ having first all coefficients of the real part and then all coefficients of the imaginary part. Thus Method II is composed of the following main steps:
Step 1 - Construct a permutation matrix $\mathbf{P}$.
Step 2 - Construct a complex-valued transformation matrix $\mathbf{M}_a$.
Step 3 - Construct $\mathbf{R}_a = \mathbf{P}^{-1}\mathbf{B}_a\mathbf{P}$, with $\mathbf{B}_a$ as defined.
The *Basic Properties* of $\mathbf{R}_a$, which is reality preserving by construction, rely on those of $\mathbf{M}_a$.
Now we take $\mathbf{M} = \mathbf{F}$ and $\mathbf{M}_a = \mathbf{F}_a = \mathbf{V}_F\mathbf{\Lambda}^a_F\mathbf{V}^t_F$ as a transform in the family obtainable by the method in [1, 2], and we obtain an

$\mathbf{R}_a$ that we can name DFrRFT.
Basic properties: *orthogonality* ($\mathbf{R}_a\mathbf{R}_a^t = \mathbf{I}$), inverse $\mathbf{R}_a^{-1} = \mathbf{R}_{-a}$, $\mathbf{R}_1 = \mathbf{P}^{-1}\mathbf{B}_1\mathbf{P}$ as the *base matrix* instead of the DFT which is not reality preserving, $\mathbf{R}_0 = \mathbf{P}^{-1}\mathbf{B}_0\mathbf{P} = \mathbf{I}$, *reality preserving* by construction, *parameter continuity* by the definition of $\mathbf{R}_a$, *parameter additivity* ($\mathbf{R}_a\mathbf{R}_b = \mathbf{P}^{-1}\mathbf{B}_a\mathbf{P}\mathbf{P}^{-1}\mathbf{B}_b\mathbf{P} = \mathbf{P}^{-1}\mathbf{B}_a\mathbf{B}_b\mathbf{P} = \mathbf{P}^{-1}\mathbf{B}_{a+b}\mathbf{P} = \mathbf{R}_{a+b}$), *parameter commutativity*, i.e. $\mathbf{R}_a\mathbf{R}_b = \mathbf{R}_b\mathbf{R}_a$, and *parameter periodicity* with $p = 4$, from $\mathbf{F}_{a+4} = \mathbf{F}_a$ [1, 2].

3.3. Decorrelation power

The decorrelation power a transform does perform is also important to know in Signal Processing. We experimentally verified that the decorrelation power of a fractional transform $\mathbf{A}_a$ varies under a in a continuous way and monotonically in $[0, 1]$.
We measure the autocorrelation of $\mathbf{A}_a\mathbf{x}$ by the *residue correlation* rc (used to compare DFT, DCT and DHT on Markov-1 signals in [15, 16]), namely $rc(\mathbf{A}_a, \mathbf{x}) = \frac{1}{N}\|\mathbf{R}_{xx} - \mathbf{A}_{-a}\mathbf{D}\mathbf{A}_a\|_H^2$, where $\mathbf{D}$ is the diagonal matrix constructed with the diagonal elements of $\mathbf{A}_a\mathbf{R}_{xx}\mathbf{A}_{-a}$, $\mathbf{R}_{xx}$ is the autocorrelation matrix of the signal $\mathbf{x}$ and $\|.\|_H^2$ is the *weak Hilbert-Schmidt norm.* $rc(\mathbf{A}_a, \mathbf{x}) \geq 0$ and 0 is achieved only in case the transform is maximally decorrelating. The *normalized residue correlation coefficient* $\chi(\mathbf{A}_a, \mathbf{x}) = \frac{rc(\mathbf{A}_a, \mathbf{x})}{\|\mathbf{R}_{xx} - \mathbf{I}_N\|_H^2}$ has a normalizing denominator measuring the maximum decorrelation amount a transform can perform.
We implemented all transforms we have here defined as *MatLab* $6.5.0.180913a\ Release\ 13$ functions and performed experiments mainly to determine their decorrelation powers. In all experiments, we ordered the eigenvalues and eigenvectors obtained by *MatLab*$6.5.0$ for our constructions of the discrete real cosine and sine transforms . We used the $MatLab$ ordering for those of the Fourier transform. We display here a figure relative to the DFrCT.2 only, with $N = 64$. We compare, in pictures (A) and (B), the residue correlation coefficients χ and rc versus $0 \leq a \leq 1$, step 0.02, on a Markov-1 signal with $\rho = 0.9$. In the interval $[-1, 0]$ those residue correlation coefficients have a symmetric behaviour. The decorrelation performance is shown with respect to increasing values of a. Pictures (C) and (D) show how rc and χ depend on the adjacent correlation ρ, $0.7 \leq \rho \leq 0.9999$, step 0.1. The decorrelation power is more significant when ρ increases, as expected. For $\rho < 0.7$, the curves have been omitted in the figures since they are less meaningful.

3.4. Conclusions and further work

The discrete fractional transforms formulated in this paper are reality preserving, have most of the required properties as well as expected monotonously decreasing decorrelation powers. They are not symmetric, since it was shown in [17] that by imposing the reality preserving property some other property has to fail. Such transforms can be useful in cases where an orthogonal, reality preserving transform is required, in which the decorrelation power is controlled by some parameter (i.e. in joint source and channel coding). Parameter additivity can also be useful in watermarking applications. Further work will be reported. Other useful results are *ad hoc* efficient algorithms (and relative software) for the new transforms. The methodology described in the paper can be used in many contexts, for example for deriving reality preserving fractional Hartley and Hadamard transforms.

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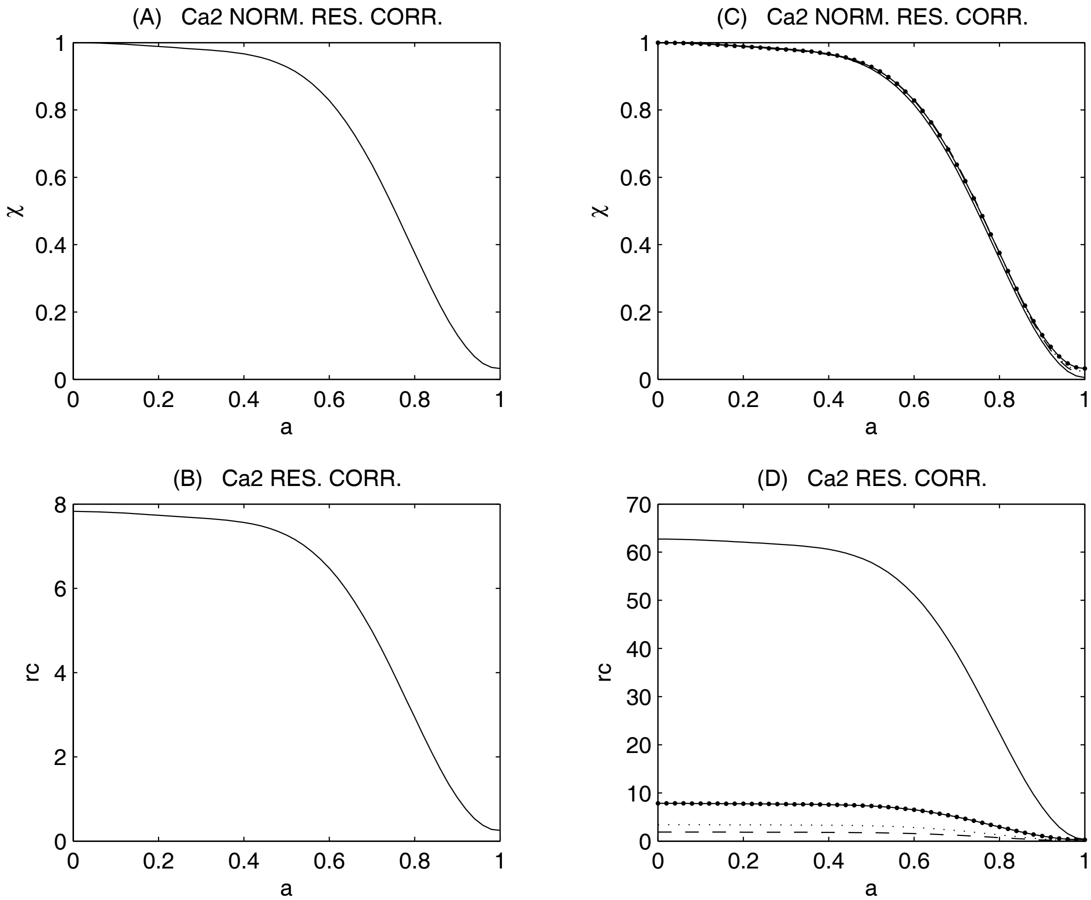

Fig. 1. Residue correlation coefficients (rc) and normalised residue correlation coefficients (χ) for the DFrCT.2 formulation, for $N = 64$, of the fractional cosine transform Ca, on Markov-1 signals. A step of 0.02 is chosen for the fractional parameter a to plot the curves of the residual correlation rc and χ in function of a. $\rho = 0.9$ in (A) and in (B). In (C) and in (D), $0.7 \leq \rho \leq 0.9999$ with step 0.1. The curve for $\rho = 0.7$ is represented with the dashed (–) line, the one for $\rho = 0.8$ with the dotted (..) line, the one for $\rho = 0.9$ with the dash-dotted (-.) line and the curve for $\rho = 0.9999$ with the solid (-) line. For $a = 0$ no decorrelation is performed, while the decorrelation power increases with a and reaches its maximum around $a = 1$. The values for $a = 1$ are near to 0, so that such a transform is quite decorrelating.