ASYMPTOTIC DISTRIBUTIONS AND PEAK POWER ANALYSIS FOR UPLINK OFDMA SIGNALS

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ABSTRACT

Statistical characterization of complex baseband signal for multicarrier multiuser systems is studied in this paper. In particular, we derive rigorously the asymptotic distribution of uplink Orthogonal Frequency Division Multiple Access (OFDMA) baseband signals for different subcarrier allocation schemes. This allows us to establish some useful statistical properties of the power process of the corresponding baseband signals. We show that in most scenarios, the power process converges to a χ^2 process. This result allows us to characterize the peak to average power ratio (PAPR) for OFDMA signal, a parameter considered critical to real system implementation. Specifically, we are able to compare, both qualitatively and quantitatively, the PAPR for different subcarrier allocation schemes. We show first that the contiguous subcarrier allocation and equally spaced interleaving share identical PAPR. Further, we show that random interleaving, which is implemented in existing OFDMA standard, has a larger PAPR than that of equally spaced interleaving scheme. The complementary cumulative distribution function of PAPR for random interleaving is shown to be greater than that of equally spaced interleaving by a factor that is identical to the number of users in the OFDMA system.

1. INTRODUCTION

Recently, Orthogonal Frequency Division Multiple Access (OFDMA) has been standardized in IEEE 802.16a for wireless Metropolitan Area Network applications. In OFDMA, all the available subcarriers are divided into mutually exclusive subchannels, each consisting of a distinct set of subcarriers. Multiple access is achieved through assigning different subchannels to different users. As such, subcarrier allocation schemes may have significant impact on the physical layer signal processing design. For example, by assuming contiguous allocation for the uplink and equally spaced interleaving for the downlink, synchronization techniques for OFDMA system have been propopsed in [1].

This work is supported in part by the CASE center of Syracuse University.

In this paper we investigate the statistical properties, and in particular, the peak-to-average power ratio (PAPR), of the uplink baseband OFDMA signal. The downlink OFDMA signal, assuming fully loaded, is equivalent to a single user OFDM signal in terms of its PAPR characterization hence existing PAPR analysis for single user OFDM systems [2,3] applies. On the other hand, the uplink OFDMA signal uses only a small fraction of subcarriers, hence is different from that of a single user OFDM signal. Specifically, we intend to address the following two questions. For different subcarrier allocation schemes, what are the asymptotic distributions of both the complex baseband signals as well as their power processes? How do different subcarrier allocation strategies affect the PAPR of the baseband signal?

Assume that there are a total of N subcarriers in an OFDMA system and they are evenly divided among Q users, thus each subchannel consists of $M = N/Q$ subcarriers, there are generally three subcarrier allocation schemes:

- Contiguous allocation (CA). The N subcarriers are divided into Q groups with each group consisting of M contiguous carriers. The first group of subcarriers is assigned to the first user and the second group allocated to the second user and so on. The subcarrier indices allocated to the kth ($0 \leq k < Q$) user are $\{kM, kM + 1, \dots, kM + M - 1\}$. See Fig. 1(a).
- Equally spaced interleaving (ESI): The N subcarriers are partitioned into M groups with each group having Q contiguous subcarriers. Then the kth subcarrier of each group is assigned to the kth user, i.e., the subcarrier indices allocated to the kth user, $0 \leq k < Q$, are $\{k, Q + k, \dots, (M - 1)Q + k\}$. See Fig. 1(b).
- Random interleaving (RI): This is adopted in IEEE 802.16a. While the subcarriers are grouped in a similar fashion as in ESI, the subcarrier index of each group assigned to a particular user is a random variable (calculated using a permutation formula in IEEE 802.16a). The subcarrier indices allocated to the kth ($0 \leq k < Q$) user are $\{\alpha_{k,1}, Q + \alpha_{k,2}, \dots, (M - 1)Q + \alpha_{k,M-1}\}$ where $\alpha'_{k,i}s$ are i.i.d. random variables with uniform distribution on $\{0, 1, \dots, Q - 1\}$.

The RI scheme has better resistance to inter-cell interference but introduces more complexity to synchronization. See Fig. 1(c).

Here we also introduce the concept of arbitrary interleaving (AI). AI differs from RI in that the subcarrier indices in each group can be arbitrary rather than having certain statistical constraints. In this case the $\alpha_{i,k}$ can take any value from $\{0, 1, \ldots, Q-1\}$ and no probabilistic model is imposed on the indices. AI can be considered as a particular realization of RI. Clearly ESI is a special case of AI.

The first part of the paper aims to characterize the statistical properties of the baseband uplink OFDMA signal for different interleaving scenarios. We have two main results:

- Result 1: The real and imaginary parts of the baseband uplink OFDMA signal converge to Gaussian processes for all subcarrier allocation schemes.
- Result 2: The power processes of the baseband uplink OFDMA signals converge to χ_2^2 processes for all subcarrier allocation schemes except for AI.

In the second part of the paper we compare the PAPR for different subcarrier allocation schemes. Based on the analysis in the first part, we have:

- Result 3: The PAPR of CA is the same as that of ESI.
- Result 4: Probabilistically, the PAPR of RI is worse than that of ESI by a factor of Q.

In the following, we use subscript c, e, r, a for CA, ESI, RI and AI. We use $\overline{x}$ and $\hat{x}$ to denote the real and imaginary parts of a complex valued x, i.e., $x = \overline{x} + j\hat{x}$

2. STATISTICAL PROPERTIES OF THE BASEBAND UPLINK OFDMA SIGNAL

For the purpose of simplicity, and without loss of generality, we consider only the 0th user in our analysis throughout this paper. It can be shown that the results derived are independent of the user index.

2.1. Baseband Signal Model

The complex baseband signal for different subcarrier allocations schemes of the uplink OFDMA can be written as

$$\begin{aligned} s_c(t) &= \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} d_k e^{j2\pi \frac{kt}{NT_s}} \\ s_e(t) &= \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} d_k e^{j2\pi \frac{kQt}{NT_s}} \\ s_r(t) &= \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} d_k e^{j2\pi \frac{(kQ+\alpha_k^R)t}{NT_s}} \\ s_a(t) &= \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} d_k e^{j2\pi \frac{(kQ+\alpha_k^A)t}{NT_s}} \end{aligned} \qquad (1)$$

for $t \in [0, NT_s]$, where NT_s is the FFT interval and α_k^R, α_k^A are the subcarrier indices in the group for RI and AI respectively. Clearly $s(t)$'s are the Fourier transforms of N symbols out of which the majority are zeros. The locations of the information bearing subcarriers depend on the particular subcarrier allocation scheme.

In the sequel, we assume that, without loss of generality, the frequency domain symbols, d_k's, are zero mean and unit variance, i.e.,

$$E(d_k) = 0, E(|d_k|^2) = 1 \qquad (2)$$

Further, the d_k's are of circular constellation, i.e.,

$$E(\overline{d}_k^2) = E(\hat{d}_k^2) = 1/2, E(\overline{d}_k \hat{d}_k) = 0 \qquad (3)$$

Note the second condition of (3) holds for most constellations including PSK, whose real and imaginary parts are dependent yet uncorrelated.

2.2. Statistical Properties of the Baseband Signal

For conventional single user OFDM system, it has been shown in [3] that the baseband signal converge weakly to a Gaussian process as the number of subcarriers goes to infinity. In this section we study the convergence of the signals for interleaved uplink OFDMA. In most cases we only state the results. For detailed proof, please see [4].

Rewrite the signal model for AI as

$$s_a(t) = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} d_k e^{j2\pi \frac{(k+\beta_k)t}{MT_s}}, t \in [0, NT_s] \qquad (4)$$

where $\beta_k = \alpha_k^A / Q$. The real and imaginary parts of $s_a(t)$ can be expressed as

$$\overline{s}_a(t) = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} \left(\overline{d}_k \cos\left(\frac{2\pi(k+\beta_k)t}{MT_s}\right) - \hat{d}_k \sin\left(\frac{2\pi(k+\beta_k)t}{MT_s}\right)\right)$$

$$\hat{s}_a(t) = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} \left(\overline{d}_k \sin\left(\frac{2\pi(k+\beta_k)t}{MT_s}\right) + \hat{d}_k \cos\left(\frac{2\pi(k+\beta_k)t}{MT_s}\right)\right)$$

We have the following theorem for the baseband signal model with arbitrary interleaving.

Theorem 1 *As $M \to \infty$, $\overline{s}_a(t)$ and $\hat{s}_a(t)$ converge to Gaussian processes in distribution.*

The following result is a direct application of Theorem 1.

Corollary 1 *As $M \to \infty$, $\overline{s}_c(t)$ and $\hat{s}_c(t)$ converge to Gaussian processes in distribution.*

Before proceeding to studying the power process of the baseband signal, we introduce the following definition [5]:

Definition 1 *If $X_1(t), \ldots, X_n(t)$ are n independent, stationary, zero mean Gaussian processes, then $Y(t) = \sum_{i=1}^{n} X_i^2(t)$ is a χ^2 process.*

Then we have the following theorem and corollary.

Theorem 2 *Denote by $A_a(t) = |s_a(t)|^2$ and $A_r(t) = |s_r(t)|^2$ the power processes of RI and AI respectively, we have*

- *As $M \to \infty$, there exist some $\{\beta_k : 0 \leq k \leq M-1\}$ when $A_a(t)$ does not converge a χ_2^2 process in distribution.*
- *As $M, Q \to \infty$, $A_r(t)$ converge to a χ_2^2 process in distribution*

Corollary 2 *For ESI and CA, $A_e(t)$, $A_c(t)$ converge to χ^2 processes in distribution as $M \to \infty$.*

To summarize, for subcarrier allocation schemes except for AI, the power process of the baseband signal converges to a χ^2 process.

3. PAPR FOR UPLINK OFDMA

Denote by η the PAPR of the baseband signal. Because the average power of the signal is unity, the PAPR is equivalent to the signal peak power, i.e.,

$$\eta = \max_{t \in [0, NT_s]} A(t) = \max_{t \in [0, NT_s]} |s(t)|^2$$

Using results derived in section 2, we compare in this section the PAPR among different subcarrier allocation schemes for uplink OFDMA signals.

3.1. A Qualitative Study

The following theorems can be derived straightforwardly.

Theorem 3 *The PAPRs for CA and ESI are the same, i.e., $\eta_c = \eta_e$.*

Sketch of proof For $\forall t_1 \in [0, NT_s]$, there $\exists t_2$ s.t. $s_c(t_2) = s_e(t_1)$ and vice versa.

Theorem 4 *As $M, Q \to \infty$, the CCDF of PAPR for RI is larger than that of ESI, i.e., $P(\eta_r > x) > P(\eta_e > x)$.*

Thus RI has, probabilistically, worse PAPR performance compared with ESI or CA. Notice that Theorem 3 is in a deterministic sense while Theorem 4 is in a statistical sense as the latter is a comparison of the CCDFs of the PAPR.

3.2. A Quantitative Study

From Theorem 4, we know that $P(\eta_r > x) > P(\eta_e > x)$. We now attempt to answer the following question: by how much is $P(\eta_r > x)$ larger than $P(\eta_e > x)$?

Since $s_e(t)$ is periodical with period MT_s, the peak power of $s_e(t)$ is itentical to the peak power during the interval $[0, MT_s]$ for $s_e(t)$. Define

$$s'_e(t) = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} d_k e^{j2\pi \frac{kt}{MT_s}} \qquad t \in [0, MT_s] \qquad (5)$$

Clearly (5) is a Fourier transform of M symbols, i.e., conventional OFDM signal with M subcarriers. The CCDF of $s'_e(t)$ can be derived as ([2]):

$$P(\eta_e > x) \approx M \sqrt{\frac{\pi}{3}} x e^{-x} \qquad (6)$$

Rewrite the baseband signal for random interleaving:

$$s_r(t) = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} d_k e^{j2\pi \frac{(k+\beta_k)t}{MT_s}} \quad t \in [0, NT_s] \qquad (7)$$

To derive the CCDF of the PAPR of $s_r(t)$, we construct a reference signal:

$$s_1(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} d'_k e^{j2\pi \frac{kt}{NT_s}} \qquad t \in [0, NT_s] \qquad (8)$$

where the input symbols $\{d'_k : 0 \leq k < N\}$ satisfy (2) and (3). It is clear that $s_1(t)$ is a conventional OFDM baseband signal with N subcarriers. The CCDF of the PAPR of $s_1(t)$ can certainly be evaluated with ease.

Consider the Nyquist rate $(1/T_s)$ sampling version of $s_r(t)$ and $s_1(t)$:

$$s_r(n) = \frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} d_k e^{j2\pi \frac{(k+\beta_k)n}{M}} \quad n = 0, \ldots, N-1$$

$$s_1(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} d'_k e^{j2\pi \frac{kn}{N}} \quad n = 0, \ldots, N-1 \qquad (9)$$

Define a complex Gaussian random variable as such that its real and imaginary parts of x are i.i.d. Gaussian, we have, for $\{s_r(n)\}$ and $\{s_1(n)\}$, the following result.

Theorem 5 *As $M, N \to \infty$, both $\{s_r(n) : n = 0, \ldots, N-1\}$ and $\{s_1(n) : n = 0, \ldots, N-1\}$ converge to a sequence of N i.i.d. complex Gaussian random variables with zero mean and unit variance.*

We now present the following conjecture:

Conjecture 1 *For two Fourier transforms $s_r(t)$ and $s_1(t)$ defined in (7) and (8), if their Nyquist rate sampled sequences (DFT) are statistically identical, then the two processes have the same peak power distribution, i.e., $P(\eta_r > x) = P(\eta_1 > x)$.*

This conjecture, though difficult to prove, is corroborated by numerical simulations. In Fig. 2 we plot the empirical CCDF of the PAPR for OFDM with $N = 1024$ and randomly interleaved OFDMA with $M = 256, Q = 4, N = 1024$ respectively. 10×10^6 OFDM (OFDMA) blocks were generated for both cases. For each OFDMA block, the α_k's are randomly generated such that they follow i.i.d. with uniform distribution in $\{0, 1, 2, 3\}$. The d_k's are i.i.d. uniformly generated from QPSK constellation. The peak of the continuousl waveform is obtained by oversampling the DFT sequence by 8 times. From the figure it can be seen that the CCDFs of the two PAPRs are very close, suggesting that Conjecture 1 holds at least for this case. Using Conjecture 1, we can approximate the CCDF of $s_r(t)$ by

$$P(\eta_r > x) \approx P(\eta_1 > x) \approx N\sqrt{\frac{\pi}{3}}xe^{-x}$$

Finally

$$\frac{P(\eta_r > x)}{P(\eta_e > x)} \approx \frac{N\sqrt{\frac{\pi}{3}}xe^{-x}}{M\sqrt{\frac{\pi}{3}}xe^{-x}} \approx Q \qquad (10)$$

In Fig. 3(a) we plot the empirical CCDFs for RI ans ESI which are consistent with that of Theorem 4. The parameter setting is the same as in Fig. 2. Fig. 3(b) further plots the empirical ratio of $CCDF_r/CCDF_e$ and it is easy to see that this ratio is very close to Q as obtained in (10).

4. CONCLUSIONS

The PAPR is a crucial limiting factor for multicarrier systems. We study in this paper the PAPR characterization for multiuser multicarrier systems. Resorting to the asymptotic regime, we derive the distributions for the uplink OFDMA baseband signal and its power process for various subcarrier allocation schemes. PAPR analysis is then carried out and we show that the RI, as is currently specified in OFDMA standard, suffers from higher PAPR as compared with other alternatives. Computer simulations are provided and are consistent with the analytical results.

5. REFERENCES

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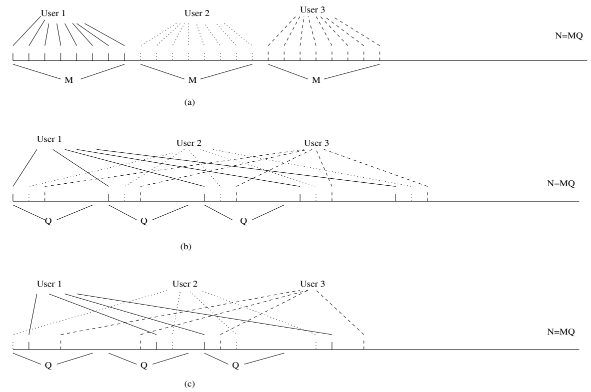


Fig. 1. 3 subcarrier allocation schemes: CA, ESI and RI

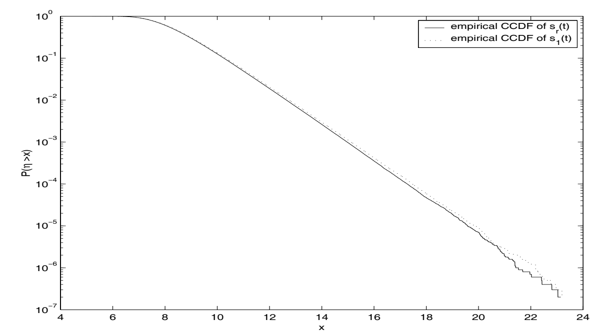
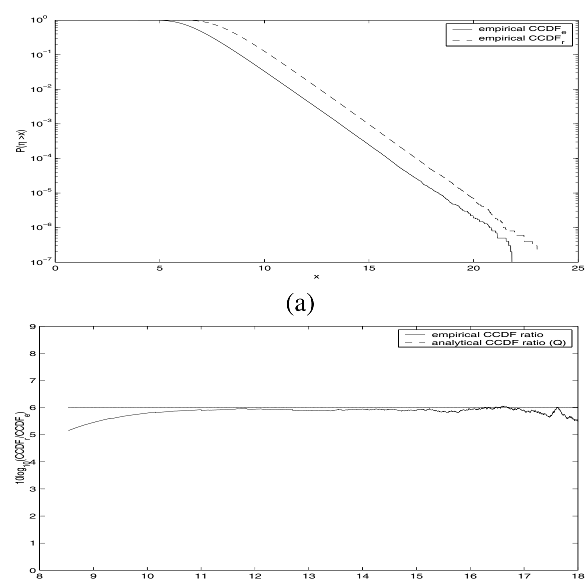


Fig. 2. The CCDFs of the PAPR for $s_r(t)$, $s_1(t)$.



(b)

Fig. 3. The CCDFs of the PAPR for RI and EI. (b) is the plot of the ratio of the CCDFs based on simulation and analytical result.