A SUBGRADIENT ALGORITHM FOR LOW COMPLEXITY DMT PAR MINIMIZATION

Alper Erdogan

EE Department, Koc University
Sariyer, Istanbul, Turkey.
email:alperdogan@ku.edu.tr

ABSTRACT

An iterative Peak-to-Average power Ratio (PAR) reduction algorithm for Discrete Multi Tone (DMT) based systems, such as OFDM and VDSL, is introduced. The proposed algorithm uses reserved or unused tones to minimize the l_∞ norm of the DMT symbol vector iteratively based on a subgradient optimization technique. The resulting iterative algorithm has a very simple update rule and therefore a low computational complexity. Furthermore, the PSD level constraints can be easily incorporated into the algorithm. The proposed algorithm's performance is illustrated for an OFDM system with 256 carriers. It is shown that a high PAR reduction is achieved especially for the cases where the PAR reduction tones are allowed to exceed the PSD mask level.

1. INTRODUCTION

Multi-carrier scheme offers various advantages especially in terms of providing an easy means of counteracting frequency selective effects of broadband channels. For this reason, it has been the choice and the candidate for several wire-line (e.g., ITU ADSL, VDSL Standards) and wireless (e.g., IEEE 802.11a) standards.

One major drawback of the multi-carrier scheme is the high effective dynamic range of the modulated signal. Considering the limitations of the analog front end in terms of its linear operation range, the high dynamic range of the DMT modulated signal causes severe challenges for the implementation.

A sizeable amount of research has been done to address this problem (see for example [1, 2, 3, 4, 5] and the references therein). The major goal has been to produce low complexity algorithms and schemes to reduce the high dynamic range to a reasonable level with no or the minimum amount of bandwidth loss. Among the existing methods, the tone reservation method proposed by Tellado[1] provides a drastic amount of reduction in the peak to average level of the multi carrier signal. The method is based on the use of the unused carriers or the carriers reserved on purpose for the reduction of the peak level of each symbol. It is shown that the optimal adjustment of the values of these tones can be formulated as a linear programming problem. Although the complexity of this linear program can be reduced by exploiting the structure of the data matrix, the resulting complexity level may still not be adequate for the real time implementation.

In this article, we propose a low complexity subgradient based tone reservation algorithm for the iterative minimization of the peak level of multi carrier signals. The resulting algorithm achieves a near-optimal peak reduction with a reasonable computational requirement. In addition, the PSD level constraints imposed by the communications standards can be easily incorporated into the algorithm.

The organization of the article is as follows: Section 2 outlines the data model and summarizes the convex optimization formulation of the reference [1] for the PAR reduction problem. Section 3 provides a brief summary of subgradient optimization algorithms. The iterative subgradient PAR reduction algorithm is provided in Section 4. In Section 5, the examples illustrating the proposed algorithm's performance are given. Finally, Section 6 is the conclusion.

2. MULTI CARRIER DATA MODEL AND PAR MINIMIZATION PROBLEM

The baseband samples for a DMT symbol is given by

$$x_n = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \frac{X_k e^{\frac{j2\pi kn}{NL}}}{\sqrt{N\cdot L}},\ n = 0,\ldots,(N+P)\cdot L-1, \quad (1)$$

where N is the FFT size(without oversampling), P is the number of prefix samples, X_k is the information signal at the k^{th} carrier and L is the oversampling factor. Let

$$\mathbf{x} = \left[\begin{array}{cccc} x_0 & x_1 & \ldots & x_{N\cdot L-1}\end{array}\right]^T \quad (2)$$

be the vector formed by the DMT symbol samples (excluding the prefix), we can write

$$\mathbf{x} = \mathbf{F}\mathbf{X} \quad (3)$$

where $\mathbf{F}$ is the IFFT matrix with

$$F_{lm} = \frac{1}{\sqrt{NL}} e^{\frac{j2\pi(l-1)(m-\frac{N}{2})}{NL}}, \quad \begin{array}{rcl} l &=& 1,\ldots,N\cdot L \\ m &=& 1,\ldots,N, \end{array} \qquad (4)$$

and

$$\mathbf{X} = \left[\begin{array}{cccccc} X_{-\frac{N}{2}+1} & \ldots & X_0 & X_1 & \ldots & X_{\frac{N}{2}} \end{array} \right]^T. \qquad (5)$$

The peak to average ratio of the corresponding DMT symbol is defined as

$$PAR = \frac{\|\mathbf{x}\|_\infty^2}{E(x_n^2)}. \qquad (6)$$

The tone reservation method proposed by [1] makes use of some reserved or unused tones to minimize the peak level of the symbol $\|\mathbf{x}\|_\infty$. If we assume that the Q tones (with the indexes $\{l_1, l_2, \ldots, l_Q\}$) are available to be used for the PAR reduction purposes, we can decompose the expression in Equation 3 as

$$\mathbf{x} = \mathbf{\Gamma}\rho + \underbrace{\mathbf{U}\varphi}_{\gamma}, \qquad (7)$$

where $\rho = \left[\begin{array}{ccc} X_{l_1} & \ldots & X_{l_Q} \end{array} \right]^T$ is the vector containing tones to be used for the PAR reduction, $\mathbf{\Gamma}$ is the matrix containing the corresponding columns of $\mathbf{F}$, and φ is the vector containing information carrying tones and $\mathbf{U}$ is the corresponding partial IFFT matrix. Based on Equation 7, the peak reduction problem can be posed as the convex optimization problem

$$\underset{\rho}{\text{minimize}} \quad \|\mathbf{x}\|_\infty = \|\mathbf{\Gamma}\rho + \gamma\|_\infty. \qquad (8)$$

As noted in [1], this can be casted as a linear programming problem. However, the conventional methods for the linear programming are not suitable for the real time implementation especially for DMT systems with high symbol rate. In [6], the alternative iterative approaches have been proposed to provide low complexity alternatives to linear programming. However, the proposed approaches try to approximate the solution of the problem in 8 rather than finding its solution.

In this article, we provide the iterative solution of the convex optimization problem in (8) using the subgradient method presented in the next section.

3. A REVIEW OF SUBGRADIENT METHODS

Let $f(\mathbf{w})$ be a convex and possibly non-differentiable function with domain S, where S is convex. The subdifferential of $f(\mathbf{w})$ at point $\mathbf{w}$ is defined as

$$\partial f(\mathbf{w}) = \{\mathbf{g} | f(\mathbf{y}) \geq f(\mathbf{w}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{w} \rangle \quad \forall \mathbf{y} \in S\}, \quad (9)$$

where $\langle \cdot, \cdot \rangle$ is the inner product. A vector $\mathbf{g}$ which is a member of $\partial f(\mathbf{w})$ is called a subgradient of $f(\mathbf{w})$ at $\mathbf{w}$. The non-differentiable counterpart of the gradient-descent algorithm is the subgradient projection method in which the gradient is simply replaced by a subgradient:

$$\mathbf{w}^{(i+1)} = \mathcal{P}_S \left\{ \mathbf{w}^{(i)} - \mu^{(i)} \mathbf{g}^{(i)} \right\} \qquad (10)$$

where $\mathbf{g}^{(i)}$ is a subgradient picked from the subdifferential set $\partial f(\mathbf{w}^{(i)})$ and $\mathcal{P}_S$ is the projection to convex set S. Although the subgradient algorithm looks very much like the gradient descent algorithm, in the subgradient iteration it may happen that $f(\mathbf{w}^{(i+1)}) > f(\mathbf{w}^{(i)})$ for any $\mu^{(i)} > 0$[7]. However, if the $\mu^{(i)}$ parameter is properly chosen, $\mathbf{w}^{(i)}$ can be made to converge to the optimal point $\mathbf{w}^*$.

One major result about the selection of the step size parameter μ_i is due to Polyak [8]: if

$$\lim_{i\to\infty} \frac{\mu^{(i)}}{||\mathbf{g}^{(i)}||} = 0 \text{ and } \sum_{i=0}^{\infty} \frac{\mu^{(i)}}{||\mathbf{g}^{(i)}||} = \infty$$

hold then $\lim_{i\to\infty} \mathbf{w}^{(i)} = \mathbf{w}^*$, which provides sufficient conditions for convergence.

Furthermore, if the step size satisfies

$$0 < \mu^{(i)} < 2\frac{(f(\mathbf{w}^{(i)}) - f^*)}{||\mathbf{g}^{(i)}||_2^2} \qquad (11)$$

where f^* is the minimum value of $f(\mathbf{w})$, then it is guaranteed that

$$||\mathbf{w}^{(i+1)} - \mathbf{w}^*||_2 \leq ||\mathbf{w}^{(i)} - \mathbf{w}^*||_2 \qquad \forall i \qquad (12)$$

i.e., the distance to the optimal vector decreases monotonically. As f^* is not known a priori in many practical problems, the use of an estimate of f^*, instead of f^* have been investigated in several references(see for example [9]). Recently Goffin and Kiwiel [10] and Sherali et. al. [11] proposed simple and convergent subgradient algorithms with variable target value $\hat{f}^*$.

4. THE SUBGRADIENT BASED PAR REDUCTION ALGORITHM

In order to find a low complexity solution to the PAR minimization problem in Equation 8, we use the subgradient approach presented in the previous section. The subdifferential set corresponding to the cost function

$$f(\rho) = \|\mathbf{\Gamma}\rho + \gamma\|_\infty \qquad (13)$$

is given by

$$\partial f(\rho) = \mathbf{Co}\left\{ \{ \frac{x_k}{|x_k|} \mathbf{\Gamma}_{k,:}^H \;\; | \;\; |x_k| = f(\rho) \} \right\}, \qquad (14)$$

where $\mathbf{Co}$ represents the convex hull operation. Based on the above subdifferential set, the PAR minimization algorithm steps can be outlined as

- Determine the time samples for which the peak level is achieved.
- The complex conjugate of the corresponding rows of $\mathbf{\Gamma}$ scaled by the magnitude normalized time samples are the subgradients.
- Possible iteration directions are the negative of the convex combinations of these subgradients.

Therefore, the possible search directions are the scaled version of the vectors which define the mapping between the frequency parameters to be adjusted and the peak values. If J is the set of time instants for which maximum magnitude is achieved, i.e., $J = \{k \quad | \quad |x_k| = ||\mathbf{x}||_\infty\}$, then a possible search direction for the subgradient projection algorithm is

$$\mathbf{d} = -\sum_{k \in J} \xi_k \frac{x_k}{|x_k|} \mathbf{\Gamma}^H_{k,:} \tag{15}$$

, where $\sum_{k \in J} \xi_k = 1$ and $\xi_k \geq 0$. For convenience, one may choose $\xi_l = 1$ for some $l \in J$ and $\xi_k = 0$ for $k \neq l$ in which case the search direction simplifies to

$$\mathbf{d} = -\frac{x_l}{|x_l|} \mathbf{\Gamma}^H_{l,:}. \tag{16}$$

As a result, we can summarize the subgradient based PAR minimization algorithm as follows

$$\rho^{(i+1)} = \rho^{(i)} - \mu^{(i)} \frac{x^{(i)}_{l^{(i)}}}{|x^{(i)}_{l^{(i)}}|} \mathbf{\Gamma}^H_{l^{(i)},:}, \tag{17}$$

where

- $l^{(i)} \in \{0, ..., N \cdot L - 1\}$ is the index where the maximum magnitude output is achieved at the i^{th} iteration.
- $\mu^{(i)}$ is the step size at the i^{th} iteration. We suggest the use of

$$\mu^{(i)} = \alpha \frac{|x^{(i)}_{l^{(i)}}| - \hat{f^*}^{(i)}}{\|\mathbf{\Gamma}_{l^{(i)},:}\|_2^2}, \tag{18}$$

as in the relaxation rule of Equation 11, where $\alpha \in [0, 2)$. Here a reasonable choice for $\hat{f^*}^{(i)}$ is given by

$$\hat{f^*}^{(i)} = |x^{(0)}_{l^{(0)}}| 10^{-\frac{\Psi}{20}}, \tag{19}$$

where Ψ is the target PAR reduction level. Alternatively, one could use the adaptive target level methods suggested in references [10, 11] to determine $\hat{f^*}^{(i)}$.

The update rule given by Equation 17 is fairly simple. Most of the computational requirement of the algorithm is due to the calculation of the contribution of the updated PAR reduction tones, i.e., $\mathbf{\Gamma}\rho$, which requires at most $N \cdot Q$ complex multiplications and N complex additions per iteration. (Since the columns of $\mathbf{\Gamma}$ are periodic complex exponentials the number of multiplications used for computing $\mathbf{\Gamma}\rho$ can be significantly reduced by exploiting this fact).

4.1. Incorporation of PSD Level Constraints

The update rule in Equation 17 assumes that the ρ can be freely selected. However, in applications, the communications standards impose some PSD mask constraints which restricts the components of ρ to go beyond certain levels.

The PSD level constraints impose

$$S = \{\rho \quad | \quad |\rho_k| \leq \beta_k \quad k = 1, \ldots, Q\} \tag{20}$$

as the feasible set of values that ρ can take, where β_k is the magnitude constraint for k^{th} component of ρ . Since S is a convex set, we can incorporate the PSD level requirements easily into our algorithm by including projection to the constraint set as suggested by Equation 10. As a result, after the update equation 17, we apply the following projection rule

$$\mathcal{P}_S\{\rho_k\} = \begin{cases} \rho_k & |\rho_k| \leq \beta_k, \\ \frac{\beta_k}{|\rho_k|}\rho_k & |\rho_k| > \beta_k, \end{cases} \tag{21}$$

to each component of ρ.

5. EXAMPLE

We simulated our algorithm for an OFDM system with 256 carriers. We reserved random 13 tones (approximately 5 percent of the tones) for PAR reduction. In simulations oversampling factor L is taken as 4 and we applied 50 iterations of the algorithm for each DMT symbol.

Figure 1 shows the results obtained from the simulation where the Complementary CDF (CCDF) functions of PAR for different cases are plotted. Solid line is for the original case where no PAR reduction scheme is applied and the dashed line represents the results of our algorithm when there are no PSD level constraints. Comparing these two curves, for a clipping level of 10^{-5} as an example, about $4.5dB$ of effective PAR reduction is achieved.

Some examples of CCDFs for the cases with different PSD level constraints are also shown in Figure 1. These sample cases are for

- the PSD Mask Level Constraint,
- 4dB Above the PSD Mask Level Constraint, and,
- 6dB Above the PSD Mask Level Constraint,

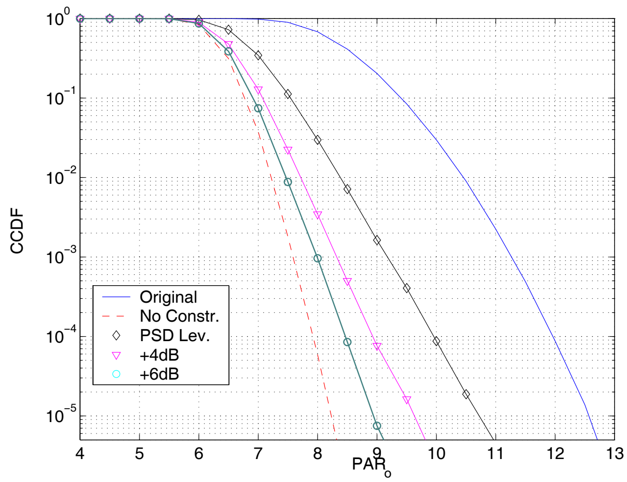

Fig. 1. PAR CCDF for different PSD level constraints.

where a flat PSD mask over all tones is assumed. Based on these curves, we can conclude that reasonable PAR reduction is achieved if the PAR reduction tones are allowed to exceed the nominal PSD level by a factor 4dB or more.

6. CONCLUSION

A low complexity l_∞ norm minimizing PAR reduction algorithm is presented. The simple subgradient expression results in a fairly simple update rule with a low computational complexity suitable for the real time implementations. Furthermore, it is easy to incorporate the PSD mask constraints into the algorithm with negligible computational cost. The simulation results show that the presented subgradient PAR reduction algorithm achieves a desirable level of performance if the PSD level constraints for the PAR reduction tones are adjusted properly.

7. REFERENCES

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