PERFORMANCE OF ZERO-FORCING EQUALIZERS FOR SINGLE-CARRIER ZERO-PADDED TRANSMISSIONS OVER MULTIPATH FADING CHANNELS

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ABSTRACT

We study bit-error rate (BER) performance of single-carrier (SC) zero-padded (ZP) transmissions with zero-forcing (ZF) equalization. Padded redundant zeros mitigate multipath effects but reduce the bandwidth efficiency. We show analytically that the BER improves as the bandwidth efficiency decreases, i.e., there is a clear trade-off between BER and bandwidth efficiency. We also demonstrate that ZP outperforms SC cyclic prefixed (CP) transmission.

I. INTRODUCTION

Severe multipath channels often arise in high-rate digital transmissions, which necessitates sophisticated equalization at the receiver. Maximum Likelihood (ML) equalization collects the available multipath diversity to improve bit-error rate (BER) performance, but is computationally cumbersome. On the other hand, linear equalization exhibits poor performance due to intersymbol interference (ISI) resulted from the multipath. Using cyclic prefix (CP) and IFFT/FFT, OFDM (orthogonal frequency division multiplexing) renders a convolution channel into parallel flat channels. A lot of success of multi-carrier transmissions based on OFDM can be found. But OFDM has several drawbacks including loss of multipath diversity gain, high peak-to-average power ratio and high sensitivity to frequency offset [5]. To overcome these disadvantages, single-carrier (SC) block transmissions have gained increasing interests recently.

In SC-CP transmissions, the CP inserted block is parallel-to-serial converted and is transmitted. Although there is no IFFT operation at the transmitter, efficient frequency domain equalization is available at the receiver [1]. With zero-forcing (ZF) equalization, SC-CP has better performance than uncoded OFDM at high SNR [4]. Another SC block transmission is zero-padded (ZP) transmission [9], where it inserts redundant zeros instead of CP into each transmitted block. Sufficient guard zeros separate two consecutive received blocks and remove interblock interference (IBI). It also leads to efficient ZF equalization with guaranteed symbol detectability regardless of the zero locations of the underlying finite impulse response (FIR) channel [7]. Guaranteed symbol detectability implies multipath diversity, and thus improved performance at moderate-high SNR [10], [11]. Indeed, ZP with ZF equalization exhibits multipath diversity at high SNR [8].

To take advantage of multipath diversity in ZP, the number of redundant symbols should be longer than the underlying channel order. In order to avoid bandwidth efficiency loss, this calls for long block sizes. But, as the block size gets long, ZP transmissions converge to conventional SC transmissions. Intuitively, the benefits of ZP would be lost for long blocks. In this paper, we analyze the relation between BER performance and bandwidth efficiency of ZP transmissions.

This paper deals with ZP transmissions with ZF equalization. We first show that for every fixed channel, its performance degrades as its block size increases. There is a clear trade-off between BER and bandwidth efficiency. This implies that at low SNR, ZP with long block size cannot take advantage of multipath diversity. But it also means that ZP outperforms conventional SC transmissions with ZF equalization, which still justifies the use of zero insertions. Numerical simulations are provided to validate our theoretical findings as well as to compare ZP with SC-CP and uncoded OFDM.

II. ZP TRANSMISSIONS

We consider point-to-point wireless transmissions over time-flat but frequency-selective fading channels. At the transmitter, the information-bearing sequence $\{s(n)\}$ is grouped into blocks $\boldsymbol{s}(n) = [s(Mn), \ldots, s(Mn+M)]^T$ of size M. To mitigate the effects of frequency selective channels, we pad M_0 zeros at the end of each block to obtain zero-padded (ZP) transmitted blocks $\{\boldsymbol{u}(n)\}$ of size $N := M + M_0$.

Our discrete-time baseband equivalent FIR channel $\{h(l)\}$ has order L, and is considered linear time-invariant. At the receiver, we assume perfect timing and carrier synchronization. We collect N noisy samples in an $N \times 1$ received vector $\boldsymbol{x}(n)$. If the number M_0 of redundant zeros is greater than or equal to the channel order L, i.e., $M_0 \geq L$, IBI is removed, and we obtain,

$$\boldsymbol{x}(n) = \boldsymbol{H}_M \boldsymbol{s}(n) + \boldsymbol{v}(n), \tag{1}$$

where $\boldsymbol{H}_M$ is a tall $N \times M$ (truncated) Toeplitz matrix with first column $[h(0), h(1), \ldots, h(L), \mathbf{0}^T]^T$ [9], and $\boldsymbol{v}(n)$ is additive white Gaussian noise (AWGN) with variance $\sigma_v^2 \boldsymbol{I}$.

Although redundant zeros reduce the bandwidth efficiency to $M/(M+M_0)$, if $M_0 \geq L$, several benefits are revealed: i) low-complexity block-by-block processing with linear ZF or minimum mean-squared error (MMSE) equalization at the receiver [9]; ii) ML equalization [10], [11] or even with ZF equalization at high SNR [8], full multipath diversity gain is enabled to enhance system performance (maximum diversity advantage); iii) blind identification of the unknown channel becomes possible [7].

With ML equalization, conventional SC transmissions without zero insertions also exhibits maximum diversity [2]. Since the bandwidth efficiency of ZP is reduced by redundant zeros, ZP with ML equalization is inferior to conventional SC transmissions in terms of bandwidth efficiency. ZP with ZF equalization also has maximum diversity at high SNR [8]. But the relation of its BER performance and bandwidth efficiency is still unclear. We will show analytically the dependency of the BER performance of ZP on its bandwidth efficiency.

III. PERFORMANCE OF ZF EQUALIZERS

We consider ZP transmissions with ZF equalization (ZF-ZP). Since at high SNR, the performance of the ZF equalization converges to that of MMSE equalization, our results below approximately hold true for MMSE equalization at high SNR.

We assume that the perfect knowledge of the channel is available at the receiver. Then, the output of a ZF equalizer can be expressed as

$$\hat{\boldsymbol{s}}(n) = \boldsymbol{s}(n) + \boldsymbol{H}_M^{\dagger} \boldsymbol{v}(n), \tag{2}$$

where $\boldsymbol{H}_M^{\dagger}$ is the pseudo-inverse of $\boldsymbol{H}_M$ defined as $\boldsymbol{H}_M^{\dagger} = (\boldsymbol{H}_M^{\mathcal{H}} \boldsymbol{H}_M)^{-1} \boldsymbol{H}_M^{\mathcal{H}}$ with $(\cdot)^{\mathcal{H}}$ denoting complex conjugate transposition.

Since $\boldsymbol{v}(n)$ is white Gaussian with variance $\sigma_v^2 \boldsymbol{I}$, the covariance of the effective noise $\boldsymbol{H}_M^{\dagger} \boldsymbol{v}(n)$ is found to be $\sigma_v^2 (\boldsymbol{H}_M^{\mathcal{H}} \boldsymbol{H}_M)^{-1}$. Since $\boldsymbol{H}_M$ is tall and has full column rank except for null channels, $(\boldsymbol{H}_M^{\mathcal{H}} \boldsymbol{H}_M)^{-1}$ always exists.

Suppose that we draw information symbols from a BPSK or a QPSK constellation and that we employ symbol-by-symbol detection based on the ZF equalized output. Let us define an $M \times M$ matrix $\boldsymbol{R}_M$ as

$$\boldsymbol{R}_M := \boldsymbol{H}_M^{\mathcal{H}} \boldsymbol{H}_M. \tag{3}$$

We denote the mth diagonal entry of $\boldsymbol{R}_M^{-1}$ as

$$\lambda_m^{(M)} := [\boldsymbol{R}_M^{-1}]_{mm}, \quad \text{for } m = 1, \ldots, M, \tag{4}$$

where $[\cdot]_{ij}$ stands for the (i,j)th entry of a matrix.

The probability of the bit-error for the mth symbol of $\boldsymbol{s}(n)$ is given by $Q([\lambda_m^{(M)}]^{-1/2} \sigma_s / \sigma_v)$ [6], where σ_s^2 is the variance of $s(n)$ and $Q(\cdot)$ is the complementary error function such that $Q(x) = 1/\sqrt{2\pi} \int_x^{\infty} e^{-t^2/2} dt$. We average the BER over one transmitted block to obtain

$$\mathrm{BER}_M := \frac{1}{M} \sum_{m=1}^{M} Q([\lambda_m^{(M)}]^{-\frac{1}{2}} \frac{\sigma_s}{\sigma_v}). \tag{5}$$

For our analysis, we utilize the following property of $\{\lambda_m^{(M)}\}$ (See Appendix for a proof):

Lemma 1: Let $\lambda_m^{(M)}$ be the mth diagonal entry of the inverse of the $M \times M$ matrix defined as (3). *Then,*

$$\lambda_1^{(M+1)} > \lambda_1^{(M)} \tag{6}$$

$$\lambda_{m+1}^{(M+1)} \geq \max[\lambda_m^{(M)}, \lambda_{m+1}^{(M)}] \quad \textit{for} \quad m = 1, \ldots M. \tag{7}$$

Since $\boldsymbol{R}_M$ is a symmetric Toeplitz matrix, it follows from $[\boldsymbol{R}_M^{-1}]_{mm} = [\text{adj } \boldsymbol{R}_M]_{mm} / |\boldsymbol{R}_M|$ that

$$\lambda_m^{(M)} = \lambda_{M-m+1, M-m+1}^{(M)}, \tag{8}$$

for $m = 1, \ldots, M$, where adj $\boldsymbol{R}_M$ and $|\boldsymbol{R}_M|$ denote the adjoint and the determinant of $\boldsymbol{R}_M$. The following lemma follows from the properties of $\{\lambda_m^{(k)}\}$:

Lemma 2: Let $f(x)$ is a monotonically increasing function in x. Then, for $\{\lambda_m^{(k)}\}$ satisfying (6), (7), *and* (8), *it holds that*

$$\frac{1}{M+1} \sum_{m=1}^{M+1} f(\lambda_m^{(M+1)}) > \frac{1}{M} \sum_{m=1}^{M} f(\lambda_m^{(M)}). \tag{9}$$

The complementary error function $Q(x)$ is a monotonically decreasing function in x. Thus, $Q([\lambda_m^{(M)}]^{-1/2} \sigma_s / \sigma_v)$ is a monotonically increasing function in $\lambda_m^{(M)}$. From Lemma 2, we can state our main result for BPSK and QPSK signaling:

Theorem 1: Suppose ZP transmissions with ZF equalization. Let the number of padded zeros be $M_0 \geq L$, where L is the channel order. Then, for every channel realization, the BER of ZP transmissions is a decreasing function in bandwidth efficiency, that is,

$$\mathrm{BER}_1 < \mathrm{BER}_2 < \cdots < \mathrm{BER}_{\infty} \tag{10}$$

where BER_M *is the BER of a ZP transmission of information block size M.*

This theorem states a deterministic property of the BER performance of ZF-ZP. The performance depends on the block size and degrades as the block size of the information-bearing symbols increases. There exists a clear trade-off between BER performance and bandwidth efficiency. One has to carefully design the block size to obtain the target BER. Theorem 1 also suggests a simple adaptive transmission scheme similar to the adaptive rate control of error correcting codes: if the target BER performance is not

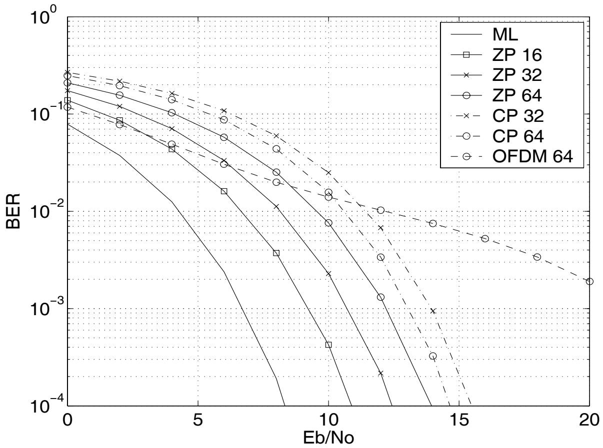

Fig. 1. BER comparison for a fixed channel

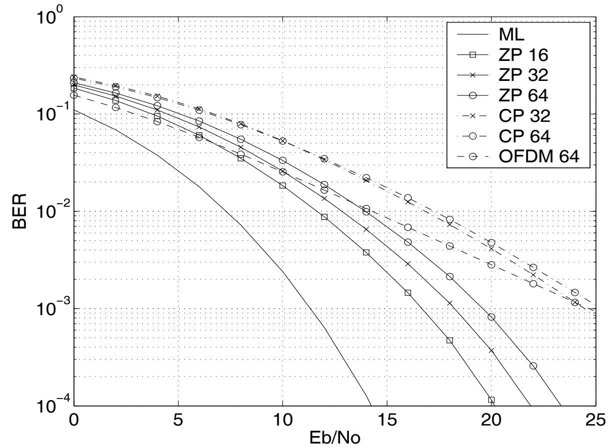

Fig. 2. BER comparison for Rayleigh channels of order 7

attained with a block size, the receiver asks the transmitter to reduce the block size to obtain a better BER.

The limit of BERs is BER_1, which is realized with $M = 1$. In this case, the BER can be expressed as $Q((\sum_{l=0}^{M} |h(l)|^2)^{\frac{1}{2}}\sigma_s/\sigma_v)$. which is identical with the BER at high SNR of ML equalization with and without zero padding [2], [10], [11]. On the other hand, as M goes to ∞, BER_M converges to $Q([\int_0^1 |H(e^{j2\pi f})|^{-2}df]^{-1/2}\sigma_s/\sigma_v)$. This is found to be equal to the BER of conventional SC transmissions with ZF equalization having (ideal) infinite length coefficients. We can conclude that ZP always outperforms conventional SC transmissions if both employ ZF equalization.

Since Theorem 1 holds true for every channel realization with order lesser than $L+1$, averaging over the channel probability density function, we establish the following:

Theorem 2: Consider ZP transmissions with ZF equalization as in Theorem 1. Then, the BER of ZP transmissions averaged over random channels is a decreasing function in bandwidth efficiency such that

$$\overline{\mathrm{BER}}_1 < \overline{\mathrm{BER}}_2 < \cdots < \overline{\mathrm{BER}}_\infty \qquad (11)$$

where $\overline{\mathrm{BER}}_M$ *denotes the average BER of a ZP transmission of information block size* M.

Suppose, for an example, i.i.d. Rayleigh channels. The average BER of ML equalization is approximated at high by $(G\sigma_s/\sigma_v)^{-(L+1)}$ with G a constant. The constant G can be viewed as a coding gain if the linear channel convolution is considered as coding over complex field [10], [11]. The slope of the BER-SNR curve, $L+1$, is the diversify order, which is resulted from i.i.d. Rayleigh channels. Although ZF-ZP has the diversity gain at high SNR [8], Theorem 2 states that ZF-ZP loses the coding gain as the bandwidth efficiency increases so that its diversity advantage could not be found at low SNR.

Unfortunately, it is not easy to obtain an explicit expression of the average BER even for i.i.d. Rayleigh channels, because the joint distribution of $\{\lambda_i^{(M)}\}$ is not available. We will resort to simulations to compare the performance of SC transmissions in the next section.

IV. NUMERICAL EXAMPLES

To validate our theoretical findings, we test ZF-ZP with different information block sizes, $M = 16, 32, 64$, for a fixed channel and for random channels. Performance of uncoded OFDM with 64 subcarriers and SC-CP transmissions [4] with blocks of sizes 32 and 64 are evaluated when hard-decoding was used at their corresponding ZF equalizer outputs.

The fixed channel is of order 3, whose coefficients are $[0.5957 + 0.0101i, -0.3273 - 0.3472i, -0.2910 - 0.0533i, 0.1285 - 0.5599i]$. BER performance of our tested systems are illustrated in Figure 1. Note that the BER performance of ML is identical with the BER of the ZF-ZP with $M = 1$. It is clear from the figure that the performance of ZF-ZP degrades as its information block size increases, which is analytically shown by Theorem 1. SC-SP does not have such a property and is inferior to ZF-ZP. It is also observed that at moderate-high SNR, ZF-ZP has better performance than the uncoded OFDM.

To verify Theorem 2, we generated 10^3 Rayleigh distributed channels of order $L = 7$, having complex zero-mean Gaussian taps with exponential power profile: $E\{|h(l)|^2\} = \exp(-l)/[\sum_l \exp(-l)]$, and averaged the results. We depict BER performance in Figure 2. As stated by Theorem 2, ZF-ZP exhibits a trade-off between bandwidth efficiency and BER performance. ZF-ZP enjoys the multipath diversity gain so that at high SNR, it outperforms uncoded OFDM, while SC-SP has the worst performance.

APPENDIX

Proof of Lemma 1: We first note that $\boldsymbol{R}_M$ has the identical structure of the correlation matrix of a moving-average process of coefficients $\{h(0),\ldots,h(L)\}$ driven by a white noise. To prove the lemma, we borrow the results on linear prediction of stationary processes. Let us consider Yule-Walker equations [3, Section 3]:

$$\boldsymbol{R}_{M+1}\begin{bmatrix} 1 \\ -\boldsymbol{a}_M \end{bmatrix} = \begin{bmatrix} P_M \\ \boldsymbol{0} \end{bmatrix}, \tag{12}$$

where $\boldsymbol{a}_M$ is the predictor vector of size M and P_M is the (forward) prediction error power.

The prediction error power of prediction order M can be expressed as $P_M = P_{M-1}(1-|\kappa_M|^2) = P_0\prod_{m=1}^{M}(1-|\kappa_m|^2)$, where κ_m is the reflection coefficient such that $|\kappa_m| \le 1$. Since $\lambda_1^{(M+1)}$ is the first diagonal entry of the inverse matrix of $\boldsymbol{R}_{M+1}$, it follows from (12) that $P_M = 1/\lambda_1^{(M+1)}$. On the other hand, the linear prediction error for moving average processes decreases as M increases, but $\kappa_m \ne 1$ for all m. Thus, (6) follows from $P_M < P_{M-1}$ and $P_M = 1/\lambda_1^{(M+1)}$.

To prove (7), we partition $\boldsymbol{R}_{M+1}$ as

$$\boldsymbol{R}_{M+1} = \left[\begin{array}{c|c} \boldsymbol{R}_M & \boldsymbol{r}_M^B \\ \hline \boldsymbol{r}_M^{B\mathcal{H}} & r \end{array}\right] = \left[\begin{array}{c|c} r & \boldsymbol{r}_M^{\mathcal{H}} \\ \hline \boldsymbol{r}_M & \boldsymbol{R}_M \end{array}\right], \tag{13}$$

where $\boldsymbol{r}_M^B$ and $\boldsymbol{r}_M$ are $M\times 1$ vectors and $r = \sum_l |h(l)|^2$. Applying the matrix inversion lemma to (13), since the forward prediction variance is identical with the backward prediction variance, we obtain

$$\boldsymbol{R}_{M+1}^{-1} = \left[\begin{array}{c|c} \boldsymbol{R}_M^{-1} & \boldsymbol{0} \\ \hline \boldsymbol{0}^T & 0 \end{array}\right] + \frac{1}{P_M}\begin{bmatrix} -\boldsymbol{b}_M \\ 1 \end{bmatrix}\begin{bmatrix} -\boldsymbol{b}_M \\ 1 \end{bmatrix}^{\mathcal{H}} \tag{14}$$

$$= \left[\begin{array}{c|c} 0 & \boldsymbol{0}^T \\ \hline \boldsymbol{0} & \boldsymbol{R}_M^{-1} \end{array}\right] + \frac{1}{P_M}\begin{bmatrix} 1 \\ -\boldsymbol{a}_M \end{bmatrix}\begin{bmatrix} 1 \\ -\boldsymbol{a}_M \end{bmatrix}^{\mathcal{H}} \tag{15}$$

where $\boldsymbol{b}_M = -\boldsymbol{R}_M^{-1}\boldsymbol{r}_M^B$ and $\boldsymbol{a}_M = -\boldsymbol{R}_M^{-1}\boldsymbol{r}_M$. Comparing diagonal entries of both sides of (14) and (15), we reach to (7).

Proof of Lemma 2: We prove the lemma only for M odd. For M even, it can be proved similarly.

Since $f(x)$ is a monotonically increasing function in x, we have from (6) and (7) that $f(\lambda_1^{(M+1)}) > f(\lambda_1^{(M)})$ and $f(\lambda_{m+1}^{(M+1)}) \ge \max[f(\lambda_m^{(M)}), f(\lambda_{m+1}^{(M)})]$ for $m = 1,\ldots M$.

Let $M = 2K+1$. Noting (8), we have

$$\frac{1}{M+1}\sum_{m=1}^{M+1} f(\lambda_m^{(M+1)}) \tag{16}$$

$$> \frac{2}{M+1}\left[f(\lambda_1^{(M)}) + \sum_{m=1}^{K}\max[f(\lambda_m^{(M)}), f(\lambda_{m+1}^{(M)})]\right].$$

On the other hand, the RHS of (9) can be expressed as

$$\frac{1}{M}\sum_{m=1}^{M} f(\lambda_m^{(M)}) = \frac{2}{M}\left[\sum_{m=1}^{K} f(\lambda_m^{(M)}) + \frac{f(\lambda_{K+1}^{(M)})}{2}\right].$$

The difference of the two BERs is lower-bounded by

$$\frac{1}{M+1}\sum_{m=1}^{M+1} f(\lambda_m^{(M+1)}) - \frac{1}{M}\sum_{m=1}^{M} f(\lambda_m^{(M)})$$

$$> \frac{2}{M(M+1)}\sum_{m=1}^{K}\Big\{M\max[f(\lambda_m^{(M)}), f(\lambda_{m+1}^{(M)})] -$$

$$\Big[mf(\lambda_m^{(M)}) + (M-m)f(\lambda_{m+1}^{(M)})\Big]\Big\}. \tag{17}$$

Since $f(x)$ is a monotonically increasing function in x, the following inequality holds true:

$$\max[f(\lambda_m^{(M)}), f(\lambda_{m+1}^{(M)})]$$

$$> \frac{m}{M}f(\lambda_m^{(M)}) + \frac{M-m}{M}f(\lambda_{m+1}^{(M)}),$$

for $m = 1,\ldots,K$. This shows that the RHS of (17) is greater than 0, which leads to (9).

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