# LINEAR MIMO PRECODERS FOR FIXED RECEIVERS

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**ABSTRACT**

We consider the problem of designing linear multiple input multiple output (MIMO) precoders for fixed receivers. We first derive a precoder that minimizes the average power subject to Signal to Interference plus Noise Ratio (SINR) constraints, and then derive a precoder that maximizes the worst case SINR subject to an average power constraint. We show that both problems can be solved using standard optimization packages. In addition, a more efficient solution based on Karush-Kuhn-Tucker (KKT) optimality conditions is presented which gives more insight into the problem. Our design promises equal SINRs and fairness among the multiple outputs. Simulation results in a multiuser system show that the proposed precoders can significantly outperform existing linear precoders.

## 1. INTRODUCTION

Many communication systems can be modelled in a MIMO setting. Recent examples that gained considerable attention are multiuser systems and communication through multiple antennas. The traditional way to deal with channel distortion and interference in such systems is receiver optimization. Recently, the search for simple, low complexity receivers, led researchers to optimize the transmitter without modifying the fixed receiver. In this paper, we propose methods for designing linear transmit precoders given a fixed effective MIMO model, which represents both the distorting channel and the suboptimal linear receiver.

One of the first results on optimizing a precoder for a fixed MIMO linear model is due to [1]. This precoder decorrelates the channel at the transmitter side. Other important contributions are reported in [1]-[7] and references within. Most of the existing precoders are based on the common minimum mean squared error (MMSE) criterion. This criterion is usually computationally attractive, but does not guarantee optimality in any of the practical performance metrics, such as Bit Error Rate (BER), throughput, or multiuser efficiency. On the other hand, in practical systems, such as systems using error correcting codes, these metrics are directly related to the output SINRs, and, in particular, to the worst SINR. Note, that this is in contrast to problems where the receiver is not fixed but jointly designed with the precoder, in which case the MMSE and SINR criteria coincide. Therefore, in this paper, we focus on SINR based precoders. In such precoders, there is a trade off between the maximal SINRs and the minimal required power. To account for both requirements, we derive two precoders. The first maximizes the minimum SINR subject to an average power constraint. The second minimizes the required average power subject to Quality of Service (QoS) constraints on the SINRs.

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Our problem formulation is very general and is applicable to any MIMO system with a fixed receiver. In particular, we examine the application of the proposed precoders in a multiuser system. We prove that, in a symmetric system, such as a system using pseudonoise (PN) sequences as signatures, the performance using the precoders with Matched Filter (MF) receivers is identical to the performance obtained by using MMSE receivers with no precoders. This result is remarkable, as it allows for each user to use a simple receiver that does not require the knowledge of all the other signatures or a matrix inversion. In non symmetric systems, the proposed precoders outperform existing precoders, and promise fairness among the different users by ensuring equal performance.

Our precoder design is based on the powerful framework of convex optimization theory [8], which allows efficient numerical solutions using standard optimization packages. In the sequel, we show that our design problems can be solved using standard conic optimization packages, such as Second Order Cone (SOC) Programming, Semi Definite Programming (SDP), or Linear Matrix Inequalities (LMI) programming. We will also establish the connection between our problem and the Generalized Eigenvalue Problem (GEVP) [9]. Moreover, using the KKT optimality conditions and Lagrange duality theory, we obtain more insight into the problems. Note that similar results were derived independently in the context of beamforming [10]-[12].

The paper is organized as follows. In Section 2 we introduce the problem formulation. Next, in Section 3, we express our design problems as standard optimization programs. In Section 4, we provide alternative solutions using the KKT conditions. Finally, in Section 5, we illustrate the use of the proposed precoders in the context of multiple user communication systems.

The following notation is used: $[\mathbf{X}]_{i,j}$ denotes the ($i$,$j$)th element of the matrix $\mathbf{X}$, $\mathrm{diag}\{x_i\}$ denotes a diagonal matrix with the elements $x_i$, $\mathrm{vec}(\mathbf{X})$ denotes the vector obtained from stacking the columns of $\mathbf{X}$, $\mathbf{e}_i$ is a vector of zeros with a one on the $i$th element, and $\mathbf{X} \succeq 0$ denotes a semipositive definite matrix $\mathbf{X}$. Finally, the operators $(\cdot)^H$, $\mathrm{Tr}\{\cdot\}$, $E[\cdot]$, $\otimes$, and $\|\cdot\|$ denote the conjugate transpose, the trace, the expectation, the Kronecker product, and the Euclidean norm, respectively.

## 2. PROBLEM FORMULATION

Consider a general, block oriented, MIMO communication system. At each time instant, a block of symbols is precoded, modulated, and transmitted over a channel. The signal at the output of the receiver can be expressed as

$$\mathbf{y} = \mathbf{H}_R\mathbf{H}_C\mathbf{H}_T\mathbf{T}\mathbf{b} + \mathbf{H}_R\mathbf{w}, \qquad (1)$$

where $\mathbf{y}$ is a length $K$ output vector, the matrices $\mathbf{T}$, $\mathbf{H}_T$, $\mathbf{H}_C$, and $\mathbf{H}_R$ represent the precoder, the transmitter, the channel, and the receiver, respectively, $\mathbf{b}$ is a length $K$ vector of independent, unit variance symbols, and $\mathbf{w}$ is a noise vector with covariance $\mathbf{R}_w$. The expected SINR at the receiver's output for the $i$'th symbol, and the expected transmitted power given by:

$$\begin{aligned} \gamma_i &= \frac{\left|[\mathbf{HT}]_{i,i}\right|^2}{\sum_{j\neq i}\left|[\mathbf{HT}]_{i,j}\right|^2+\sigma_i^2}, \qquad i=1,\cdots,K; \\ P &= \mathrm{Tr}\left\{\mathbf{T}^H\mathbf{H}_T^H\mathbf{H}_T\mathbf{T}\right\}, \end{aligned} \tag{2}$$

where $\mathbf{H}=\mathbf{H}_R\mathbf{H}_C\mathbf{H}_T$ is the effective MIMO channel, and $\sigma_i^2 = \left[\mathbf{H}_R\mathbf{R}_w\mathbf{H}_R^H\right]_{i,i}$.

It is well known that the performance metrics of a communication system (BER, throughput, etc.) are highly related to its SINRs, and, in particular, are dominated by the worst SINR, namely, the smallest SINR. A conflicting performance metric is the average transmitted power. Thus, we propose two opposite criteria for the design. The first designs the precoder to maximize the minimum SINR, subject to a power constraint:

$$\mathrm{PCO}(P_o):\left\{\begin{array}{lll} \max_{\mathbf{T},\gamma_o} & \gamma_o & \\ \text{s.t.} & \gamma_i\geq\gamma_o, & i=1,\cdots,K; \\ & P\leq P_o. & \end{array}\right. \tag{3}$$

The second criterion designs the precoder to minimize the required power subject to SINR constraints[1]:

$$\mathrm{SCO}(\gamma_o):\left\{\begin{array}{lll} \min_{\mathbf{T},P_o} & P_o & \\ \text{s.t.} & \gamma_i\geq\gamma_o, & i=1,\cdots,K; \\ & P\leq P_o. & \end{array}\right. \tag{4}$$

Optimizations PCO and SCO are closely related. The only difference is that in PCO the parameter $P_o$ is fixed and $\gamma_o$ is optimized, whereas in SCO the parameter $\gamma_o$ is fixed and $P_o$ is optimized. It is easy to see that the optimal $\gamma_o$ of PCO is continuous, and strictly monotonically increasing in $P_o$, and that the optimal $P_o$ of SCO is continuous, and strictly monotonically increasing in $\gamma_o$. Furthermore, if $\gamma_o$ is optimal for PCO($P_o$), then $P_o$ is optimal for SCO($\gamma_o$), and vice versa. An interesting and attractive property of the optimization problems is that at the optimal solution of both problems all the constraints are active, i.e., the designs promise equal SINRs and fairness among all the outputs.

## 3. STANDARD SOLUTIONS

In this section, we show that the two design problems can be represented as standard optimization programs. Thus, both can be efficiently solved using standard optimization packages.

### 3.1. SCO program

It can be shown that the SCO program can be formulated as:

$$\begin{array}{lll} \min_{\mathbf{T},\sqrt{P_o}} & \sqrt{P_o} & \\ \text{s.t.} & \sqrt{\gamma_o^{-1}}\,[\mathbf{HT}]_{i,i}\geq\left\|\mathbf{t}_{\lfloor i\rfloor}\right\|, & i=1,\cdots,K; \\ & \sqrt{P_o}\geq\left\|\mathrm{vec}(\mathbf{H}_T\mathbf{T})\right\|, & \end{array} \tag{5}$$

where

$$\mathbf{t}_{\lfloor i\rfloor} = \left[\begin{array}{c}\left(\mathbf{I}-\mathbf{e}_i\mathbf{e}_i^H\right)\mathbf{T}^H\mathbf{H}^H\mathbf{e}_i \\ \sigma_i\end{array}\right], \quad i=1,\cdots,K. \tag{6}$$

[1] In general, each output can be constrained to a different value, but in this paper we consider the case of equal values.

The two constraints in (5) are of the form $z_1(\mathbf{x})\geq\| \ \boldsymbol{z}_2(\mathbf{x})\ \|$, where the scalar $z_1(\mathbf{x})$ and the vector $\boldsymbol{z}_2(\mathbf{x})$ depend affinely on the optimization variables $\mathbf{x}$. Such inequalities define convex sets which are called SOC. Thus, the program in (5) can be solved using any standard SOC package [13].

Moreover, any SOC can also be represented as an LMI, i.e., a cone obeying $\mathbf{Z}(\mathbf{x})\succeq 0$, where the matrix $\mathbf{Z}(\mathbf{x})$ depends affinely on the optimization variables $\mathbf{x}$. For example, the SINR inequalities are also equivalent to

$$\left[\begin{array}{cc}\sqrt{\gamma_o^{-1}}\,[\mathbf{HT}]_{i,i} & \mathbf{t}_{\lfloor i\rfloor}^H \\ \mathbf{t}_{\lfloor i\rfloor} & \sqrt{\gamma_o^{-1}}\,[\mathbf{HT}]_{i,i}\,\mathbf{I}\end{array}\right]\succeq 0, \quad i=1,\cdots,K. \tag{7}$$

Thus, the SCO program can also be solved using standard LMI packages [14]. However, SOC solvers have a much better worst case computational complexity than LMI solvers for this problem.

### 3.2. PCO program

Let us now turn to the PCO program. At first glance, it seems similar to SCO. However, it turns out to be considerably more complicated. This is because the matrix inequalities in (7) are linear in $\beta_o=\sqrt{\gamma_o^{-1}}$ or in $\mathbf{T}$, but not in both simultaneously. Thus, when $\beta$ is an optimization variable and not a parameter, these constraints are no longer LMIs. In fact, the sets which they define are not convex[2]. Nonetheless, if we rewrite (7) and separate out the terms which are linear, we have

$$\beta\overbrace{\left[\begin{array}{cc}[\mathbf{HT}]_{i,i} & 0 \\ 0 & [\mathbf{HT}]_{i,i}\mathbf{I}\end{array}\right]}^{\mathbf{A}_i(\mathbf{T})}\succeq\overbrace{\left[\begin{array}{cc}0 & -\mathbf{t}_{\lfloor i\rfloor}^H \\ -\mathbf{t}_{\lfloor i\rfloor} & \mathbf{0}\end{array}\right]}^{\mathbf{A}_i(\mathbf{T})}, \tag{8}$$

where $\mathbf{A}_i\left(\mathbf{T}\right)$ and $\mathbf{B}_i\left(\mathbf{T}\right)$ are matrices that depend affinely on $\mathbf{T}$. Using (8) we can express PCO as

$$\begin{array}{lll} \min_{\mathbf{T},\beta_o} & \beta_o & \\ \text{s.t.} & \beta_o\mathbf{A}_i\left(\mathbf{T}\right)\succeq\mathbf{B}_i\left(\mathbf{T}\right), & i=1,\cdots,K; \\ & \mathbf{P}\leq P_o. & \end{array} \tag{9}$$

Although not convex, problems with the structure in (8) have been investigated in the context of control theory, and are known as GEVP, i.e., minimizing the maximum generalized eigenvalue of a pencil of matrices $\mathbf{A}_i\left(\mathbf{T}\right)$ and $\mathbf{B}_i\left(\mathbf{T}\right)$ that depend affinely on the optimization variables (for more details see [9] and references within). Such problems can be solved using appropriate software, e.g., the GEVP command in the LMI toolbox [14].

A different approach for solving PCO, that does not require a dedicated GEVP software, exploits the connection between the PCO program and the SCO program. Specifically, we can solve PCO($P_o$) by iteratively solving its convex counterpart $\widehat{P}_o$ =SCO($\widehat{\gamma}_o$) for different $\widehat{\gamma}_o$'s until we find a solution in which $\widehat{P}_o=P_o$:

[2] The exact definition of such sets is quasi convex [8].

PCO($P_o$)
$\gamma_{\max} \leftarrow \text{MaxSINR}$
$\gamma_{\min} \leftarrow \text{MinSINR}$
**repeat**
  $\gamma_o \leftarrow (\gamma_{\min} + \gamma_{\max})/2$
  $[\mathbf{T}, \widehat{P}_o] \leftarrow \text{SCO}(\gamma_o)$
  **if** $\widehat{P}_o \leq P_o$
    **then** $\gamma_{\min} \leftarrow \gamma_o$
    **else** $\gamma_{\max} \leftarrow \gamma_o$
**until** $\widehat{P}_o = P_o$
**return** $\mathbf{T}, \gamma_o$

Due to the strict monotonicity, the algorithm will converge.

## 4. KKT BASED SOLUTIONS

In the previous section we showed that the PCO and SCO programs can be solved efficiently using standard optimization packages. To reduce the computational complexity of the algorithms, and obtain more insight into the problems, we present alternative solutions based on the KKT conditions.

### 4.1. SCO program

We can develop a solution to SCO by solving the necessary and sufficient KKT conditions for optimality, e.g., [8]. Tedious algebraic manipulations result in the following solution:

$$\mathbf{T} = \left[\mathbf{H}^H \mathbf{\Lambda} \mathbf{H} + \mathbf{H}_T^H \mathbf{H}_T\right]^{-1} \mathbf{H}^H \text{diag}\left\{\alpha_i\right\}, \quad (10)$$

$$P_o = \sum_i \lambda_i \sigma_i^2, \quad (11)$$

where $\mathbf{\Lambda} = \text{diag}\{\lambda_i\}$, and

$$\alpha_i^2 = \sum_k \left[\left(\left(1 + \frac{1}{\gamma_o}\right) \text{diag}\left\{[\mathbf{F}]_{i,i}\right\} - \mathbf{F}\right)^{-1}\right]_{i,k} \sigma_k^2, \quad (12)$$

$$[\mathbf{F}]_{i,j} = \left|\mathbf{H}\left[\mathbf{H}^H \mathbf{\Lambda} \mathbf{H} + \mathbf{H}_T^H \mathbf{H}_T\right]^{-1} \mathbf{H}^H\right|_{i,j}^2, \quad i,j = 1, \cdots, K,$$

and the dual parameters $\lambda_i \geq 0$ satisfy

$$\lambda_i \left[\mathbf{H}\left(\mathbf{H}^H \mathbf{\Lambda} \mathbf{H} + \mathbf{H}_T^H \mathbf{H}_T\right)^{-1} \mathbf{H}^H\right]_{i,i} = \frac{\gamma_o}{1 + \gamma_o}, \quad (13)$$

for $i = 1, \cdots, K$. Thus, to determine the optimal solution $\mathbf{T}$ and $P_o$, we need to find the variables $\lambda_i$ that are the solution to (13). We now present two methods for finding them. The first method solves (13) using the following fixed point iteration:

$$\lambda_i^{(n+1)} = \frac{\gamma_o}{1 + \gamma_o} \frac{1}{\left[\mathbf{H}\left(\mathbf{H}^H \mathbf{\Lambda}^{(n)} \mathbf{H} + \mathbf{H}_T^H \mathbf{H}_T\right)^{-1} \mathbf{H}^H\right]_{i,i}}, \quad (14)$$

for $i = 1, \cdots, K$. It can be shown that this iteration is a generalization of the beamforming algorithm in [11] to the problem of beamforming, and that convergence is promised. Following [10], an alternative approach for finding $\lambda_i$ is:

$$\begin{aligned} &\max_{\lambda_i \geq 0} \textstyle\sum_i \lambda_i \sigma_i^2 \\ &\text{s.t.} \quad \mathbf{H}^H \mathbf{\Lambda} \mathbf{H} + \mathbf{H}_{\text{Tx}}^H \mathbf{H}_{\text{Tx}} - \left(1 + \tfrac{1}{\gamma_o}\right) \lambda_i \mathbf{H}^H \mathbf{e}_i \mathbf{e}_i^H \mathbf{H} \succeq 0, \quad (15) \\ &\qquad i = 1, \cdots, K. \end{aligned}$$

This problem is concave and can be easily solved using an LMI optimization package. Note that it is more efficient than the solution in (5) as it has only $K$ optimization variables, rather than $K^2 + 1$. On the other hand, it requires an LMI software, and therefore is less appealing than the fixed point iteration in (14).

### 4.2. PCO program

The PCO program can also be solved using the previous KKT conditions. As explained, the optimal solution of PCO is also optimal for an inverse SCO program. Thus it must fulfill the previous conditions too. The only difference is that, because $\gamma_o$ is unknown, we need to find a different scaling for $\lambda_i$ that will satisfy (11). A simple iteration in this case is

$$\begin{aligned} \widetilde{\lambda}_i &= \frac{1}{\left[\mathbf{H}\left(\mathbf{H}^H \mathbf{\Lambda}^{(n)} \mathbf{H} + \mathbf{H}_T^H \mathbf{H}_T\right)^{-1} \mathbf{H}^H\right]_{i,i}}; \\ \lambda_i^{(n+1)} &= \frac{\widetilde{\lambda}_i}{\sum_i \sigma_i^2 \widetilde{\lambda}_i} P_o, \quad i = 1, \cdots, K \end{aligned} \quad (16)$$

Once the iteration converges, the optimal $\mathbf{T}$ is given by (10).

### 4.3. Symmetric case

The KKT conditions in (13) allow a simple closed form solution for the symmetric case. In this case, the matrix $\mathbf{H}$ has equal diagonal elements, equal off diagonal elements, and $\sigma_i^2 = \sigma^2$. Due to the symmetry it is clear that choosing $\lambda_i = \frac{P}{K\sigma^2}$ will satisfy (13). Therefore, the optimal solution is

$$\begin{aligned} \mathbf{T} &= c\left[\mathbf{H}^H \mathbf{H} + \frac{K\sigma^2}{P_o} \mathbf{H}_T^H \mathbf{H}_T\right]^{-1} \mathbf{H}^H, \\ \gamma_o &= \frac{1}{\frac{1}{\left[\mathbf{H}\left(\mathbf{H}^H \mathbf{H} + \frac{K\sigma^2}{P_o} \mathbf{H}_T^H \mathbf{H}_T\right)^{-1} \mathbf{H}^H\right]_{i,i}} - 1}, \end{aligned} \quad (17)$$

where $c$ is a constant that scales the matrix to satisfy the power constraint. An identical precoder was previously proposed in [4] and [5] using an MMSE design criterion. However, it maximizes the SINR only in the symmetric case, and not in general. Note, that the well known decorrelator of [1] is a good approximation for (17) when $P_o \gg K\sigma^2$.

## 5. APPLICATION TO MULTIUSER SYSTEMS

In this section, we present an application of the proposed precoders to a multiuser downlink system. At each symbol's period the base station transmits $K$ symbols using an $N \times K$ signature matrix $\mathbf{H}_T = \mathbf{S}$. The signatures are normalized so that $\left[\mathbf{S}^H \mathbf{S}\right]_{i,i} = 1$, and the cross correlations are denoted by $\left[\mathbf{S}^H \mathbf{S}\right]_{i,j} = \rho_{i,j}$. We assume an ideal channel $\mathbf{H}_C = \mathbf{I}$ and equal noise variances $\sigma^2$. Each user detects its symbols using one of the standard linear receivers:

- MF receiver, $\mathbf{H}_R = \mathbf{S}^H$.
- Decorrelator, $\mathbf{H}_R = \left(\mathbf{S}^H \mathbf{S}\right)^{-1} \mathbf{S}^H$.
- MMSE receiver, $\mathbf{H}_R = \left(\mathbf{S}^H \mathbf{S} + \sigma^2 \mathbf{I}\right)^{-1} \mathbf{S}^H$.

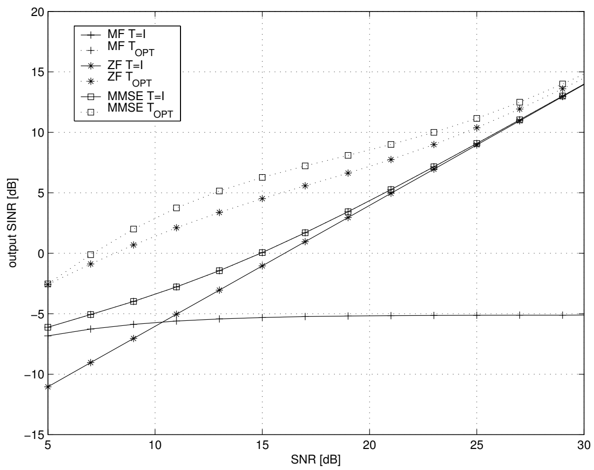


**Fig. 1**. SINR of a symmetric 3 users system.

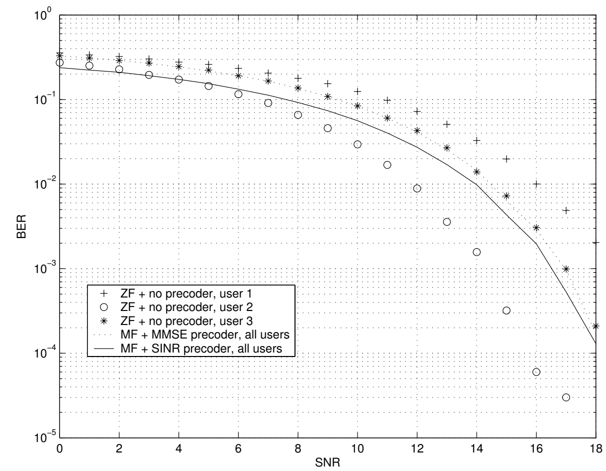


**Fig. 2**. BERs of a non symmetric 3 users system.

We first consider the symmetric case in which $\rho_{i,j} = 0.9$. In Fig. 1 we plot the output SINRs given by (17) for the three linear receivers. For comparison, we also plot the output SINRs that result from similar systems without a precoder. From the figure, we see that our precoder using a MF receiver attains the performance of an MMSE receiver without a precoder. Moreover, when our precoder is used with a ZF or an MMSE receiver, the output SINRs improve even more.

As a second example, we consider an equal power system with unequal cross correlations between the users signatures. In such systems, there is no closed form expression for the performance. Therefore, we resort to Monte Carlo simulations. Following [1], we simulate the cross correlations $\rho_{12} = 0.8$, $\rho_{13} = 0.9$, and $\rho_{23} = 0.7$. Each user uses an MF receiver. For comparison, we provide BER results of the decorrelator precoder [1], and the PCO precoder. In addition, a system without a precoder using a ZF receiver is also examined. The results are provided in Fig. 2. Due to the asymmetry, each of the three users performs differently without the precoders. On the other hand, the precoders promise fairness and equal BERs for all the users. Naturally, the performance of the best user degrades, but this is less important from a system prospective because the overall performance is dominated by the worst user. When compared to the decorrelator, the PCO precoder gains up to 1dB.

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