# PERFORMANCE OF DIVERSITY RECEPTION OVER FADING CHANNELS WITH IMPULSIVE NOISE

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**ABSTRACT**

We derive the performance of post-detection-combining (PDC) over Rayleigh fading channels with impulsive noise and compare it with maximum-ratio-combining (MRC). The bit error rates (BER) and simple upper and lower bounds are derived in closed form for the first time in the literature for these two techniques over impulsive noise channels. We show how the impulsive noise deteriorates the performance compared with Gaussian noise. PDC is shown to be more robust to impulsive noise than MRC, especially when the number of antenna is large. Simulation results corroborate our analysis.

## 1. INTRODUCTION

Diversity is one of the most important tools for combating the deleterious effects of fading in wireless channels. Exploiting diversity necessitates having access to multiple copies of the transmitted signal. Depending on the complexity and the degree of channel knowledge at the receiver, different diversity combining techniques have been proposed in the literature which include Maximal Ratio Combining (MRC), Equal Gain Combining (EGC), Selection Combining (SC), and Post-Detection Combining (PDC) [1]. The noise in each diversity branch for these schemes have often been assumed Gaussian. However, man-made electromagnetic interference, atmospheric noise, or ignition noise are prevalent, and often impulsive [12, 8]. This requires analysis and design of diversity combining schemes by taking into account the impulsive nature of the noise.

The design of optimum receivers over fading, impulsive noise channels were considered in [4, 3], where a spherically invariant random process was assumed for the noise distribution. Adaptive diversity receivers were considered in [2] wherein the receiver adapts to the unknown parameters of the noise process. Performance over nonfading impulsive noise channels for optimum and suboptimum receivers were analyzed in [6], and iterative algorithms for decoding codes over the complex field were introduced in [7]. Studies that analyze the performance of DPSK with EGC and SC over Ricean fading channels were considered in [10] and in [13].

Our aim in this paper is to analyze the performance of conventional receivers: MRC and PDC over impulsive noise channels. Unlike [10] and [13], we will consider coherent combining schemes and derive simple bounds that give insight about how much the impulsive nature of the noise deteriorates the performance compared to the Gaussian case, over fading channels. We will also prove that the post-detection combiner is particularly robust to impulsive noise, due to its nonlinear nature. This was observed in [2], but the BER performance was not derived. In fact, we theoretically show that the BER upper bound of PDC does not depend on the impulsive noise parameters, while the upper bound of MRC does. Recalling that the PDC is considerably less complex than MRC, it appears that PDC is a good diversity combining technique in impulsive environments.

In Section 2 we adopt Middleton's Class A model for the noise distribution. In Section 3, we analyze the performance of MRC and PDC under the impulsive noise model and derive upper and lower bounds. In Section 4 the numerical results are presented, and Section 5 concludes the paper.

## 2. SYSTEM MODEL

Consider the following $L^{th}$ order diversity reception model:

$$x_l = \sqrt{\rho}\; h_l s + w_l \;, l = 1, \ldots, L \;, \tag{1}$$

where $x_l$ is the received signal, $h_l$ is a complex Gaussian channel gain with zero-mean and variance normalized to 1, $w_l$ is the noise term corresponding to the $l^{th}$ diversity branch, $s \in \{-1/\sqrt{2}, 1/\sqrt{2}\}$ is the transmitted symbol from a BPSK constellation, $\rho$ is the average SNR per branch, and we have dropped the time index since we assume symbol-by-symbol decoding. We assume that each noise sample $w_l := g_l + i_l$ is the superposition of a background Gaussian component $g_l$, and impulsive component $i_l$ with $T := \mathrm{var}(g_l)/\mathrm{var}(i_l)$, denoting their power ratio. We will assume Middleton's class-A model wherein the pdf of the complex valued noise at any one of the branches can be written as:

$$p(w_l) = \sum_{m=0}^{\infty} \frac{\alpha_m}{\pi\sigma_m^2} \exp\left(-\frac{|w_l|^2}{\sigma_m^2}\right) \;, \tag{2}$$

where $\alpha_m := \exp(-A)A^m/m!$, $\sigma_m^2 := \sigma^2(m/A + T)/(1+T)$, and $\sigma^2 := \mathrm{var}(w_l)$. As defined before, $T$ represents the power ratio of the background noise and the impulsive part, and $A$ is the so-called impulsive index, which would yield an impulsive $i_l$ for small values of $A$, and a near Gaussian $i_l$ , when $A$ is large [12, 8, 6]. As clearly seen from its pdf in (2) the noise $w_l$ is not Gaussian. However, the class-A noise can be viewed as *conditionally* Gaussian, also referred to as compound Gaussian. Therefore,

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$w_l$, when conditioned on a Poisson random variable $P_l$ with parameter $A$, is Gaussian with mean zero and variance

$$v_l := \mathrm{var}(w_l|P_l) = \sigma^2 \left( \frac{P_l}{A(T+1)} + \frac{T}{T+1} \right),\ l = 1, ..., L. \tag{3}$$

Note that $\mathrm{var}(w_l) = E_{P_l}[\mathrm{var}(w_l|P_l)] = E_{P_l}[v_l] = \sigma^2$, where the last equality follows from (3) and the fact that $E[P_l] = A$. The integer random variable $P_l$ can be interpreted as the state of the noise indicating whether there is an impulse present (in which case $P_l > 0$) and the strength of the impulsive noise. To complete our description of the noise model, we have to specify the joint distribution of $v_1, \ldots, v_L$, which will determine the joint distribution of $w_l$. we assume that $v_l$ are i.i.d. random variables with distribution as described in (3), so that the joint distribution of $\mathbf{w} := [w_1, \ldots, w_L]$ is given by $p(\mathbf{w}) = \prod_{l=1}^{L} p(w_l)$. Clearly the $w_1, \ldots, w_L$ are statistically independent [5].

In this paper, we will investigate the performance of diversity combining schemes under the above noise model. The channel state information $h_1, \ldots, h_L$ is assumed known or estimated at the receiver, however the Poisson *noise state* $P_1, \ldots, P_L$, are assumed *unknown* at the receiver. We will focus on the performance for BPSK over Rayleigh fading channels. The extension to M-PSK or M-QAM with Ricean or Nakagami fading channels is straightforward, but not included here because of space limitations.

Throughout we will use the following vector notation for the random variables involved: $\mathbf{P} := [P_1, \ldots, P_L]$, and $\mathbf{h} := [h_1, ..., h_L]$. Also we use $E_{A,B}[C]$ to denote the expected value of the random variable $C$ with respect to the distributions of the random variables $A$ and $B$.

## 3. PERFORMANCE OF MRC AND PDC

In this section, we will analyze the performance of MRC and PDC for impulsive channels and derive simple performance bounds.

### 3.1. MRC Performance

We will start with the performance of the MRC receiver. The conventional MRC receiver decides on the transmitted signal by computing

$$\hat{s} = \mathrm{sign} \left( \sum_{l=1}^{L} \Re\{h_l^* x_l\} \right) , \tag{4}$$

where $\Re\{\cdot\}$ denotes the real part, and $\mathrm{sign}(\cdot)$ is 1 for positive arguments and 0 else. We note that (4) maximizes the SNR at the decision device, however, since $w_l$ are not Gaussian, maximization of the SNR does not translate into minimization of the BER. We will still analyze the performance of conventional MRC receiver in (4) to see how robust it is to impulsive noise. The probability of error of MRC conditioned on the channel $\mathbf{h}$ and any realization of the noise state $\mathbf{P}$ can be written as

$$\mathrm{BER}_{MRC}(\rho|\mathbf{h}, \mathbf{P}) = Q \left( \sqrt{ \frac{2\rho \left( \sum_{l=1}^{L} |h_l|^2 \right)^2}{\sum_{l=1}^{L} |h_l|^2 v_l} } \right) , \tag{5}$$

by expressing the conditional instantaneous SNR at the output of the combiner in terms of $\rho$, $|h_l|$, and $v_l$. The average bit error rate is given by taking expectation of (5) with respect to the distributions of $\mathbf{h}$ and $\mathbf{P}$, which requires the computation of an $L$-fold integral and sum. Unfortunately this is not analytically tractable. The best avenue for obtaining exact expressions for the BER is performing the expectations using Monte Carlo integration. However, since this does not provide insight as to how the average BER depends on the noise parameters, we will proceed with an upper bound for MRC.

To find the upper bound, we start by using the standard bound on the $Q(\cdot)$ function to bound (5):

$$\mathrm{BER}_{MRC}(\rho|\mathbf{h}, \mathbf{P}) \leq \frac{1}{2} \exp \left( - \frac{\rho \left( \sum_{l=1}^{L} |h_l|^2 \right)^2}{\sum_{l=1}^{L} |h_l|^2 v_l} \right) . \tag{6}$$

Suppose now that $v_{max} := \max_l v_l$. Then it is easy to see that the right hand side of (6) can be further upper bounded by replacing each $v_l$ in the denominator with $v_{max}$ and canceling out the common $\sum_{l=1}^{L} |h_l|^2$ from the numerator and denominator:

$$\mathrm{BER}_{MRC}(\rho|\mathbf{h}, \mathbf{P}) \leq \frac{1}{2} \exp \left( - \frac{\rho \sum_{l=1}^{L} |h_l|^2}{v_{max}} \right) . \tag{7}$$

Using the fact that each $|h_l|^2$ is exponentially distributed, we can average both sides of (7) with respect to $\mathbf{h}$:

$$E_{\mathbf{h}} \left[ \mathrm{BER}_{MRC}(\rho|\mathbf{h}, v_{max}) \right] \leq \frac{1}{2} \left( \frac{v_{max}}{v_{max} + \rho} \right)^L . \tag{8}$$

Taking expectations of both sides in (8) with respect to $v_{max}$ we get

$$E_{\mathbf{h},\mathbf{P}} \left[ \mathrm{BER}_{MRC}(\rho|\mathbf{h}, \mathbf{P}) \right] \leq \frac{1}{2} \sum_{m=0}^{\infty} P[v_{max} = \sigma_m^2] \left( \frac{\sigma_m^2}{\sigma_m^2 + \rho} \right)^L . \tag{9}$$

To calculate (9) we need an expression for $P[v_{max} = \sigma_m^2]$. Recalling that $v_{max}$ is the maximum of $L$ iid random variables given in (3), its probability mass function is easily computed as:

$$P[v_{max} = \sigma_m^2] = \left( \sum_{k=0}^{m} e^{-A} \frac{A^k}{k!} \right)^L - \left( \sum_{k=0}^{m-1} e^{-A} \frac{A^k}{k!} \right)^L . \tag{10}$$

Thus, the upper bound is given by substituting (10) into (9).

It is possible to considerably simplify (9) by sacrificing the tightness of the bound. To show how, we write (10) as $F_\nu^L(m) - F_\nu^L(m-1)$ where $F_\nu(m) := P[\nu \leq \sigma_m^2]$. By expanding (10) using binomial theorem, and using the fact that $F_\nu(m) \leq 1$, we can show that

$$P[v_{max} = \sigma_m^2] \leq LP[v = \sigma_m^2] = L\alpha_m . \tag{11}$$

We can now further upper bound the right hand side in (9) by discarding the $\sigma_m^2$ term in the denominator yielding

$$\begin{aligned} E_{\mathbf{h},v_{max}} \left[ \mathrm{BER}_{MRC}(\rho|\mathbf{h}, v_{max}) \right] &\leq \frac{1}{2} \sum_{m=0}^{\infty} P[v_{max} = \sigma_m^2] \left( \frac{\sigma_m^2}{\rho} \right)^L \\ &\leq \left( \frac{\rho}{\sigma^2} \right)^{-L} \left( \frac{L}{2} \sum_{m=0}^{\infty} \frac{e^{-A} A^m (1+T)}{m! \left( \frac{m}{A} + T \right)} \right), \end{aligned} \tag{12}$$

where for the last inequality we used (11). This upper bound is loose when SNR is small, and it is a tight bound at high SNR, which actually reflects the diversity order of MRC, as we will show in Fig. 1. In the following, we will investigate the performance of the PDC.

### 3.2. Post-Detection Combining

The PDC makes hard decisions on each branch (by looking at their sign), and combines these decisions:

$$\hat{s} = \sum_{l=1}^{L} \text{sign}\left[\Re\left(h_l^* x_l\right)\right] \ . \tag{13}$$

Notice that the nonlinear operation of taking the sign is applied before adding the contributions of the branches in this case, as opposed to what we saw for MRC in (4). This is well-known to reduce the diversity order by a factor of two over the nonimpulsive Gaussian channel. The loss in diversity order will also be present when the noise is impulsive, however, we will derive a bound on the performance of (13) that is independent of $A$, showing that (13) is robust to impulsive noise. Also, PDC has the merit of easy implementation for both coherent and non-coherent detection, which motivates its use for both Gaussian and impulsive noise channels. We first begin by deriving the exact BER and then proceed with the bound. For any realization of the random variables $h_l$ and $v_l$, the probability of making an incorrect decision at the $l^{th}$ branch is given by

$$p_l := Q\left(\sqrt{\frac{2\rho|h_l|^2}{v_l}}\right) \ . \tag{14}$$

Making an error after combining the branches as in (13) means having $L/2$ or more branches be incorrect. Hence, for a given realization of $\mathbf{h}$ and $\mathbf{P}$, the BER is the probability that more than $L/2$ branches or more are in error. This probability is given by

$$\text{BER}_{PDC}(\rho|\mathbf{h},\mathbf{P}) = \sum_{k=\lceil \frac{L}{2} \rceil}^{L} \sum_{\mathcal{S}_k \subset \mathcal{L}} \prod_{i \in \mathcal{S}_k} p_i \prod_{j \in (\mathcal{L}-\mathcal{S}_k)} (1-p_j) \tag{15}$$

where $\mathcal{L} := \{1,\ldots,L\}$, and $\mathcal{S}_k$ is any subset of $\mathcal{L}$ with $k$ elements, and $-$ is the set difference operation. Hence the inner sum in (15) is over all subsets of $\mathcal{L}$ with size $k$, and contains $\binom{L}{k}$ terms. Since $v_l$ and $h_l$ are iid, so are $p_l, l = 1,\ldots,L$. Then, the expected value of each term in the inner sum with respect to $p_l$ is the same and given by $p^k(1-p)^{L-k}$, where $p := E_{h_l,v_l}[p_l]$. This means that the expected value of (15) reduces to
$E_{\mathbf{h},\mathbf{P}}[\text{BER}_{PDC}(\rho|\mathbf{h},\mathbf{P})] =$

$$\begin{cases} \sum_{k=\lceil \frac{L}{2} \rceil}^{L} \binom{L}{k} p^k (1-p)^{L-k}, & \text{L odd;} \\ \sum_{k=\lceil \frac{L}{2} \rceil+1}^{L} \binom{L}{k} p^k (1-p)^{L-k} + \frac{1}{2}\binom{L}{\frac{L}{2}} p^{\frac{L}{2}} (1-p)^{\frac{L}{2}}, & \text{L even.} \end{cases} \tag{16}$$

where $p := E_{h_l,v_l}[p_l]$. In order to determine the average bit error rate, the only thing that remains is the computation of $p$. But this can be done by averaging (14) with respect to $h_l$ and $v_l$ respectively. Averaging (14) with respect to $h_l$ we obtain the well-known expression [11]:

$$E_{h_l}[p_l] = \frac{1}{2}\left(1 - \sqrt{\frac{\rho}{v_l+\rho}}\right) \ . \tag{17}$$

Averaging (17) with respect to $v_l$, we obtain

$$p = \frac{1}{2}\sum_{m=0}^{\infty} \alpha_m \left(1 - \sqrt{\frac{\rho}{\sigma_m^2+\rho}}\right) \ . \tag{18}$$

Hence, the exact expected bit error rate of the sign detection scheme is given by (16) where $p$ is given by (18). Clearly, this closed-form expression is useful in calculating the performance for any value of $\rho$, $A$, $\sigma^2$ and $T$. However, it is possible to upper bound (16) to obtain a simpler formula that illuminates some of the properties of the PDC in the presence of impulsive noise. To derive this upper bound, we will first notice that

$$\sum_{k=\lceil \frac{L}{2} \rceil}^{L} \binom{L}{k} p^k (1-p)^{L-k} \leq \sum_{k=\lceil \frac{L}{2} \rceil}^{L} \binom{L}{k} p^k \ , \tag{19}$$

because $0 \leq (1-p)^{L-k} \leq 1$. We then proceed to derive an upper bound on $p$ that will give us the desired simple expression. Using (14), and the standard bound on the $Q(\cdot)$ function we have $p_l \leq (1/2)\exp(-\rho|h_l|^2/v_l)$. Taking expectations of both sides with respect to $h_l$, we obtain

$$E_{h_l}[p_l] \leq \frac{1}{2(1+\frac{\rho}{v_l})} \ . \tag{20}$$

Removing the additive 1 term will only make (20) larger, yielding $E_{h_l}[p_l] \leq v_l/(2\rho)$. Taking expectation of both sides with respect to $v_l$ we obtain

$$p = E_{h_l,v_l}[p_l] \leq \frac{E[v_l]}{2\rho} = \left(\frac{2\rho}{\sigma^2}\right)^{-1} \ . \tag{21}$$

Substituting (21) into the right hand side of (19) we obtain the following bound:

$$E_{\mathbf{h},\mathbf{P}}[\text{BER}_{PDC}(\rho|\mathbf{h},\mathbf{P})] \leq \sum_{k=\lceil \frac{L}{2} \rceil}^{L} 2^{-k} \binom{L}{k} \left(\frac{\rho}{\sigma^2}\right)^{-k} \ . \tag{22}$$

In addition to being simpler than the exact expression, the bound in (22) sheds light on a very important aspect of the PDC: the right hand side of (22) does not depend on $A$ or $T$. This means no matter what values $A$ and $T$ take (i.e., no matter how impulsive the noise is), the average BER will always be bounded by (22). This is in contrast with the performance of the MRC receiver, which was sensitive to the impulsive index $A$, and could perform poorly for small values of $A$. Another important point is that since $k \geq L/2$ in (22), the diversity order of the sign receiver is seen from (22) to be at least $L/2$. This is worse than a diversity of order $L$ for MRC. Here, we see that the performance of diversity reception over non-Gaussian impulsive noise channels show that there is a tradeoff between the attainable diversity and coding gains.

## 4. NUMERICAL RESULTS

In this section, we will compare the performance of MRC and PDC by Monte-Carlo simulations, and our theoretical results. We choose two different sets of $A$ and $T$. One is a more impulsive noise channel with $A = 10^{-4}$ and $T = 0.1$, and the other is more Gaussian with $A = 1$ and $T = 0.1$. We consider $L = 2$ and $L = 4$.

In Fig. 1, we show the BER of PDC and MRC over the more impulsive channel ($A = 10^{-4}, T = 0.1$) when $L = 2$ and $L = 4$. We also show the upper bound of PDC and MRC (we only show the upper bound for MRC when SNR $>$ 30dB since it is loose at low SNR). From this figure we see that the performance of MRC is deteriorated by the impulsive noise: the BER curve is almost flat from 10 - 35dB. Only at the SNR range of 35 - 50dB, we can observe the diversity of $L$ is achieved.

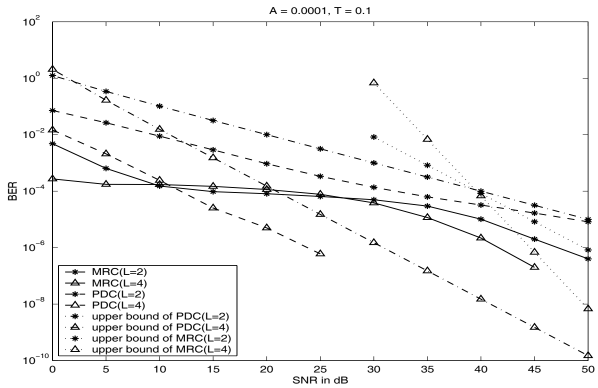


**Fig. 1**. MRC and PDC over Rayleigh Fading Channels with Impulsive Noise: $A = 10^{-4}, T = 0.1$

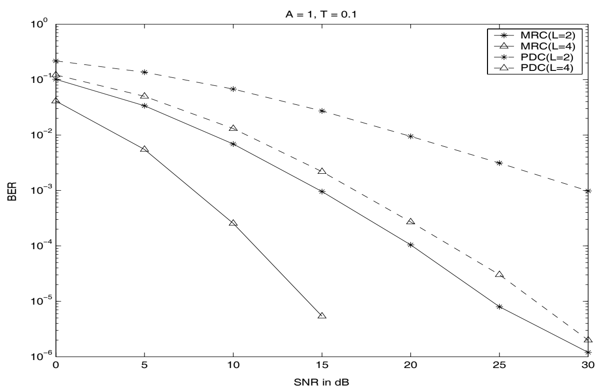


**Fig. 2**. MRC and PDC over Rayleigh Fading Channels with Impulsive Noise: $A = 1, T = 0.1$

The BER curve of PDC is different from that of MRC. It goes down continuously with SNR increases. Also, the diversity can be seen to be approximately $L/2$, which is in accordance with our analysis. The upper bound given in (22) is tight when SNR is high.

When $L = 2$, we see that MRC outperforms PDC for all SNRs. However, when $L = 4$, the PDC outperforms MRC when SNR is between 10 - 45dB as shown in Fig. 1. It is reasonable to conjecture that MRC outperforms PDC at a very high SNR due to the higher diversity order of MRC. We do not consider the extremely high SNR here since it is not practical. The advantage of PDC over MRC with more antennas is intuitive, since PDC first makes decisions over each antenna, and then collect the decisions and make a final decision at the receiver. If a large impulsive noise occurs at a few channels, they will affect the final decision of MRC. However, in PDC, making decisions at each antenna first and then combining the decisions may eliminate the effect of the wrong decisions on these channels, if the number of wrong decisions are less than $L/2$. So PDC can in fact combat large impulsive noise especially when $L$ is large.

If the noise is more Gaussian ($A = 1, T = 0.1$), we see from Fig. 2 that MRC outperforms PDC for all $L$. It is also clear from the plot that the diversity order of MRC is $L$ and that of PDC is $L/2$. This result is in accordance of the traditional analysis of the performance over Rayleigh fading channel with AWGN.

From our simulation results, we may conclude that for a more Gaussian channel, MRC is better than PDC. However, for a more impulsive channel, when $L \geq 4$, PDC is better than MRC.

## 5. CONCLUSIONS

In this paper, we discuss the performance of PDC over Rayleigh fading channels with impulsive noise and compare it with MRC. We analyze the BERs and gives upper and lower bounds. Through our analysis and simulation results, we conclude that the MRC, which is optimal over Gaussian noise is not necessarily optimal in a the fading channel with impulsive noise. Instead, PDC is more robust to impulsive noise at a moderate SNR especially when $L$ is large. Hence PDC is a good candidate for diversity combining in highly impulsive environments.

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