Frequency Synchronization in OFDM

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***Abstract*— We present an OFDM frequency synchronization scheme. The scheme uses periodic OFDM symbols, similarly to the algorithms proposed previously by Morelli and Mengali [8] and Schmidl and Cox [6]. The proposed scheme attains considerably higher accuracy than the scheme by Schmidl and Cox requiring a similar computational load. Compared to the scheme by Morelli and Mengali, the proposed algorithm attains a somewhat inferior accuracy but at a significantly reduced computational complexity, *i.e* $O(N)$ versus $O(N^2)$ operations for N-tone OFDM. In addition to that, the scheme proposed here is considerably less sensitive to the accuracy of the involved computations than the other two schemes.**

I. Introduction

Orthogonal frequency division multiplexing is a communication technique with a long history [1], [2], [3] which is rapidly emerging as a technology of choice in wireless applications. International standards such as IEEE 802.11 are employing OFDM for wireless LAN and other nomadic applications. For mobility applications, OFDM is also a contender for being the technology of choice for fourth generation systems. For wireless applications, an OFDM-based system can be of particular interest because it provides a greater immunity to impulse noise and fast fades and facilitates equalization, while efficient hardware implementations can be realized using FFT techniques. The emergence of OFDM has also motivated a revival of research activities on various implementation issues including the problem of time and frequency estimation.

The problem of frequency synchronization for OFDM has been extensively studied in the literature. We focus on methods that estimate frequency offset using a pilot sequence. In this direction, a number of ideas have been previously proposed by Moose [4], Van de Beek *et al.* [5], Schmidl and Cox [6], Müller-Weinfurtner [7], Morelli and Mengali [8], and Song *et al.* [9] and others. These schemes provide various trade-offs between performance and complexity. For instance, the method of Schmidl and Cox is very simple to implement but performs far from the theoretical limits. In contrast, Morelli and Mengali propose an algorithm the accuracy of which is very close to theoretical limits; that is, however, paid by a considerable increase in computational complexity. Other implementation issues such as robustness to computation errors and quantization effects are also of importance when designing an OFDM physical layer.

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In this paper, we present a new frequency synchronization scheme for OFDM that attains a close-to-optimal performance at a low computational complexity. The new scheme is also more robust to computational inaccuracies and quantization errors than the schemes proposed by Schmidl and Cox [6] and Morelli and Mengali [8].

II. The System Model

In OFDM, the coded input symbol sequence $a[i]$, $i = 0, 1, \ldots$, is the input to a serial to parallel device with N outputs. At each time $l = 0, 1, 2, \ldots$, the output $a_k[l]$ at port $k = 0, 1, \cdots, N-1$ is given by $a_k[l] = a[lN + k]$. The output of the serial-to-parallel multiplexer is the input to an IFFT device whose output at time l is given by

$$b_l[n] = (1/N) \sum_{k=0}^{N-1} a_k[l] e^{j2\pi kn/N} \ , \ 0 \le n < N \ . \qquad (1)$$

The output of the IFFT device is then put into serial form using a parallel-to-serial multiplexer to which then a cyclic prefix of length G is appended. Thus, the sequence $s_l[n], -G \le n < N$ that is the output of the cyclic extension device is given by: $s_l[n] = b_l[N+n], -G \le n < 0$, $s_l[n] = b_l[n], 0 \le n < N$. Windowing is then performed and the resulting sequence is input to a digital-to-analog converter and the transmit chain. At the receiver, the received signal is first down-converted and at this point a coarse time and frequency synchronization is performed, following which the output is digitized and the cyclic extension is removed. Following serial-to-parallel conversion, an FFT operation is performed and the output is put back into serial form. This produces the input to the decoder whose function is to recover the transmitted bit sequence.

If the cyclic prefix is longer than the channel impulse response and time/frequency synchronization is perfect, the intersymbol interference in time domain is completely eliminated. However, due to the poor frequency localization of the modulating complex exponentials, even relatively minor frequency offset may cause considerable inter-carrier interference.

Assuming lack of perfect frequency synchronization, the received sequence is given by

$$r_l[n] = e^{j2\pi(f_0 n_\epsilon + f_\epsilon n + f_\epsilon n_\epsilon)} \sum_{i=0}^{L-1} s_l[n-i] h_i + w_n \ , \ 0 \le n < N \ , \qquad (2)$$

where f_0 and f_ϵ are the carrier frequency and the frequency offset, respectively, normalized by the sampling frequency, n_ϵ

is the timing mismatch normalized by the sampling interval, $h_0, h_1, \ldots, h_{L-1}$ is the impulse response of the channel, and w_n is the additive noise. Given that coarse time and frequency synchronization is performed, it can be assumed that $n_\epsilon < 1$.

III. Frequency Synchronization

In non-blind schemes, estimates of the frequency offset f_ϵ and the timing mismatch n_ϵ are evaluated from the observed vector $r_n,\ 0 \le n < N$, as given in (2), which corresponds to a known pilot sequence $s_n,\ 0 \le n < N$.[1]

Consider using a periodic test sequence of the form

$$s_{nK+i} = s_i\ ,\ 0 \le i < K\ ,\ 0 \le n < N/K\ .$$

Assuming that the cyclic prefix extension is longer that the channel impulse response, the corresponding received sequence is also periodic (apart from the noise component) and is given by:

$$r_{nK+i} = t_i e^{j(\phi_i + 2\pi f_\epsilon (nK+i))} + w_{nK+i} \qquad (3)$$
$$0 \le i \le K-1\ ,\ 0 \le n \le N/K - 1\ ,$$

where $t_i = \left|\sum_{k=0}^{L-1} s_{i-k} h_k\right|$, and $\phi_i = 2\pi(f_0 n_\epsilon + f_\epsilon n_\epsilon) + \arg\left(\sum_{k=0}^{L-1} s_{i-k} h_k\right)$. Cramer-Rao bound for frequency offset estimation from such a received sequence can be show to be

$$E(|f_\epsilon - \hat{f}_\epsilon|^2) \ge 3\left(2\pi^2 \mathrm{snr} N(N-K)(N+K)\right)^{-1}\ , \qquad (4)$$

where $\mathrm{snr} = \left(\sum_{i=0}^{K-1} t_i^2\right)/K\sigma^2$. Shorter periods K attain lower bounds and vice versa. The best estimation performance is achieved with $K = 1$, however, this case is disadvantageous in terms of frame synchronization, so one needs to consider $K \ge 2$. In the other extreme case, $K = N$, the corresponding Fisher information matrix becomes singular.

Frequency offset f_ϵ and phase angles $\phi_0, \ldots, \phi_{K-1}$ can be determined by minimizing the square error between the phase angles of the observed symbols, $\arg(r_{nK+i}) = \alpha_{nK+i}$, and their respective values $\arg(r_{nK+i}) = \phi_i + 2\pi f_\epsilon(nK + i)$ in the case when there is no noise in the channel. The corresponding error function is given by

$$\mathcal{E} = \sum_{i=0}^{K-1} \sum_{n=0}^{N/K-1} (\alpha_{nK+i} - \phi_i - 2\pi f_\epsilon(nK+i))^2\ . \qquad (5)$$

The minimization of this error function with respect to f_ϵ and $\phi_0, \phi_1, \ldots, \phi_{K-1}$ yields our frequency offset estimator:

$$\hat{f}_\epsilon = C_1 \sum_{i=0}^{K-1} \sum_{n=0}^{N/K-1} (-N + K + 2Kn)\alpha_{Kn+i}\ , \qquad (6)$$

where $C_1 = 3/\pi N(N-K)(N+K)$. Note that the expected value of α_{nK+i} can be shown to be

$$E(\alpha_{nK+i}) = E(\arg(r_{nK+i})) = \phi_i + 2\pi f_\epsilon(nK+i)\ , \qquad (7)$$

[1]From now on, we omit frame index l and use r_n and s_n to denote $r_l[n]$ and $s_l[n]$, respectively.

hence, the proposed procedure yields unbiased estimates of the frequency offset and the phase angles ϕ_i. Observe also that f_ϵ is determined from (5) by estimating the average slope of the increase of α_{nK+i} in n and that slope is equal to $2\pi K f_\epsilon$. The range of the proposed frequency offset estimator is therefore $f_\epsilon < \pi/K$, which is sufficient for state-of-the-art oscillators and considered OFDM sampling frequencies. However, phase angles α_{nK+i} may fall outside of the range $(-\pi, \pi)$ due to their evolution caused by the frequency offset or due to the additive noise. This phase warping effect degrades the performance of the proposed algorithm and we discuss some methods of dealing with it in the implementation section of the paper.

IV. Performance Analysis

The remaining error of the proposed frequency offset estimator has the form

$$f_\epsilon - \hat{f}_\epsilon = C_1 \sum_{n=0}^{N/K-1} (-N + K + 2Kn) \sum_{i=0}^{K-1} \delta_{Kn+i}\ , \qquad (8)$$

where δ_n are zero-mean noise components. It follows that the variance of this estimator is

$$E(|f_\epsilon - \hat{f}_\epsilon|^2) = 3\delta^2\left(\pi^2 N(N-K)(N+K)\right)^{-1}\ , \qquad (9)$$

where $\delta^2 = E(|\delta_n|^2)$.

This expression is valid for any uncorrelated zero-mean perturbation δ_n, and we will use it also to assess the effects of computational errors on the estimation accuracy. If δ_n are noise components, their variance can be found by observing that

$$\delta_n = \arctan\left(\frac{\left|\frac{w_n}{t_n}\right| \sin\left(\arg(\frac{w_n}{t_n})\right)}{1 + \left|\frac{w_n}{t_n}\right| \cos\left(\arg(\frac{w_n}{t_n})\right)}\right)\ . \qquad (10)$$

From this expression it can be further shown that $E(|\delta_n|^2) = 1/(2\mathrm{snr})$, which finally gives

$$E(|f_\epsilon - \hat{f}_\epsilon|^2) = 3/\left(2\pi^2 N(N-K)(N+K)\mathrm{snr}\right)^{-1}\ . \qquad (11)$$

Hence, our estimator is asymptotically efficient.

The proposed synchronization algorithm, as well as previously studied frequency offset estimators, involve calculation of the argument of a complex number. Assuming that the quantization errors made in determining angles α_{Kn+i} in expression (6) are zero-mean and independent, the variance of the frequency estimator due to the quantization error is also given by formula (8); in particular

$$E(|f_\epsilon - \hat{f}_\epsilon|^2) = 3\Delta^2\left(\pi^2 N(N-K)(N+K)\right)^{-1}\ , \qquad (12)$$

where Δ^2 is the variance of the quantization error.

Schmidl and Cox [6] propose using a test sequence s_n such that $s_{n+N/2} = s_n$. Hence, the phase of the product $r_n r^*_{n+N/2}$ satisfies $\arg(r^*_n r_{n+N/2}) = N\pi f_\epsilon + \delta$, where δ is the noise component. The frequency offset is then estimated in two steps: first, the component which corresponds the fractional part of Nf_ϵ is determined as

$$\hat{f}_\epsilon = (\pi N)^{-1} \arg \sum_{n=0}^{N/2-1} r_n^* r_{n+N/2} \ , \tag{13}$$

then, the component which corresponds to the integral part of Nf_ϵ is determined using one additional test sequence and calculating a few more correlations between sequences of length $N/2$. The estimator by Schmidl and Cox already at low signal-to-noise ratios approaches closely the error function given by $E(|f_\epsilon - \hat{f}_\epsilon|^2) = 6(\pi^2 \mathrm{snr} N^3)^{-1}$. The expression on the right-hand side of this equation is equal to the Cramer-Rao bound for frequency offset estimation using one test sequence which consists of two identical halves. That Cramer Rao bound can be obtained from (4) by taking $K = N/2$. One can observe that the estimation accuracy can be improved by 1.25dB if sequences composed of shorter identical segments (K small compared to N) are used. Note, however, that Cramer-Rao bound for frequency offset estimation using two sequences of length N is 10.25dB below the accuracy attained by Schmidl-Cox method, and that a fair comparison between this method and methods which do not require two training sequences should consider the accuracy of the latter methods attained using training sequences of length $2N$. Our method, therefore, has the potential for improving the accuracy over the method by Schmidl and Cox by 10.25dB for the same amount of training data. In the following section we report simulation results which show a 9dB improvement. The discrepancy between the expected and attained improvements is due to imperfect phase unwarping which occurs in implementations of our algorithm.

Using training sequences which consist of more than two identical segments has been previously proposed by Morelli and Mengali [8]. Their scheme also requires only one training sequence and estimates frequency offset from correlations:

$$R(m) = \frac{1}{N - mK} \sum_{k=mK}^{N-1} r_k r^*(k - mK) \ , \ 0 \le m \le N/2K \ . \tag{14}$$

The algorithm proposed by Morelli and Mengali is also asymptotically efficient, *i.e.* its accuracy attains Cramer-Rao bound in (4) as the signal-to-noise ratio increases. However, their algorithm is computationally considerably more complex than the algorithm by Schmidl and Cox or our algorithm. In particular, the method by Morelli-Mengali requires $O(N^2)$ complex multiplications and additions for computing the correlations in (14).

Let us now compare the three algorithms in terms of their sensitivity to the quantization involved in evaluating the argument of a complex number. In the case of Schmidl-Cox estimator, it follows immediately from formula (13) that the quantization error of variance Δ^2 induces an estimation error the variance of which is $E(|f_\epsilon - \hat{f}_\epsilon|^2) = \Delta^2(\pi^2 N^2)^{-1}$. On the other hand, it can be shown that in the case of the method proposed by Morelli and Mengali the variance of the estimation error caused by this quantization is approximately $E(|f_\epsilon - \hat{f}_\epsilon|^2) = \Delta^2(7.5\pi^2 N)^{-1}$. Hence, the algorithm proposed in this paper reduces the error caused by the quantization of $\arg(\cdot)$ function by factor $N/3$ compared to the method by Schmidl and Cox, and by a factor close to $N^2/25$ compared to the scheme proposed by Morelli and Mengali.

V. Implementation and Simulation Results

The frequency offset estimator proposed here finds the frequency offset as the average, over i, of the slopes of the straight lines which best approximate observed phase angle sequences $\{\alpha_i, \alpha_{i+K}, \alpha_{i+2K}, \ldots, \alpha_{i+N-K}\}$ (see expression (6)). In absence of any noise, these observed angles should be $\alpha_{i+nK} = \phi_i + 2\pi f_\epsilon(nK + i)$. The angles α_j are determined using $\arctan$ function, $\alpha_j = \arctan(\mathrm{Im}(r_j)/\mathrm{Re}(r_j)$, which causes phase warping if an actual α_j falls outside of the range $(-\pi, \pi)$. The proposed frequency offset estimator, therefore, requires an additional phase unwarping algorithm.

The first step of the phase unwarping algorithm which we used in the simulations reported here consists of filtering each sequence $\{\alpha_i, \alpha_{i+K}, ..., \alpha_{i+N-K}\}$, $i = 0, 1, ..., K-1$, using filter $g = [1\ 1\ 1\ 1\ 1\ -1\ -1\ -1\ -1\ -1]$. The length of this filter is not critical. By observing relative positions and values of maxima in the filtered sequences, we determine those α_{i+nK} where phase warping occurs. In addition, based on these maxima we can determine whether a certain α_{i+n_pK} is the beginning of phase warping at all subsequent α_{i+nK}, $n > n_p$, or if α_{i+n_pK} is a point of an isolated phase warping caused by additive noise.

From unwarped phase angle sequences we find estimates $\hat{f}_\epsilon^{\,i}$, $i = 0, ..., K-1$, of the frequency offset as

$$\hat{f}_\epsilon^{\,i} = 4C_1 \sum_{n=0}^{N/K-1} (-N + K + 2Kn)\alpha_{nK+i} \ . \tag{15}$$

These estimates are then relabeled as $\hat{f}_\epsilon^{\,i_k}$, $k = 0, ..., K-1$, in the following manner. Recall that $\hat{f}_\epsilon^{\,i_k}$ is obtained from subsequence

$$r_{i_k+nK} = t_{i_k} e^{j(\phi_{i_k} + 2\pi f_\epsilon (nK+i_k))} + w_{i_k+nK}, \ n = 0, 1, ...$$

We label $\hat{f}_\epsilon^{\,i_k}$ so that $t_{i_0} \ge t_{i_1} \ge ... \ge t_{i_{K-1}}$. The magnitude parameters t_i can be also estimated by minimizing the error function in (5) which gives $\hat{t}_i = (K/N) \sum_{n=0}^{N/K-1} |r_{nK+i}|$. Finally we consider the following set of estimators,

$$f_k = \mathrm{mean}(\hat{f}_\epsilon^{\,i_0}, \hat{f}_\epsilon^{\,i_1}, ..., \hat{f}_\epsilon^{\,i_k}) \ , \ k = 0, 1, ..., K-1 \ ,$$

and select for the final frequency offset estimator that f_k which minimizes the mean square error between a straight line with the slope $2\pi K f_k$ and the corresponding sequence

$$\mathrm{mean}(\alpha_{i_0+nK}, \alpha_{i_1+nK}, ..., \alpha_{i_k+nK}) \ , \ n = 0, ..., N/K-1 \ .$$

The rationale behind this final selection process is that we believe that the phase unwarping would be most accurate for those sequences $\{\alpha_i, \alpha_{i+K}, ..., \alpha_{i+N-K}\}$ which correspond to largest values of t_i. Hence, first the estimate $f_0 = \hat{f}_\epsilon^{\,i_0}$ is obtained from $\{\alpha_{i_0}, \alpha_{i_0+K}, ..., \alpha_{i_0+N-K}\}$. Then this estimate is refined as $f_1 = \mathrm{mean}(\hat{f}_\epsilon^{\,i_0}, \hat{f}_\epsilon^{\,i_1})$ if the error of approximating the sequence

$$\left\{\frac{\alpha_{i_0}+\alpha_{i_1}}{2}, \frac{\alpha_{i_0+K}+\alpha_{i_1+K}}{2}, ..., \frac{\alpha_{i_0+N-K}+\alpha_{i_1+N-K}}{2}\right\}$$

using a straight line with slope f_1 is smaller than the error of approximating $\{\alpha_{i_0}, \alpha_{i_0+K}, ..., \alpha_{i_0+N-K}\}$ using a line with slope f_0, and so on.

Simulation results which compare our algorithm with the algorithms proposed by Schmidl and Cox and by Morelli and Mengali are shown in Figures 1 and 2. Simulations are performed for $N = 128$-tone OFDM, assuming a three-tap Rayleigh fading channel, five-point cyclic prefix, and frequency offset $f_\epsilon = 0.001$. We obtained identical results for $f_\epsilon = 0.02$. In all simulations parameter K was taken to be $K = 4$. In Figure 1, the curve denoted by SC2 represents the method by Schmidl and Cox that requires two training sequences of length N. The curves labeled CT1 and CT2 on the same figure represent results obtained using our algorithm with one or two training sequences, respectively. The curve obtained using the algorithm by Morelli and Mengali is not shown in Figure 1; it is almost indistinguishable from the Cramer-Rao bound which corresponds to the two-training-sequence case, plotted by the dash-dot line. The curve labeled CT2/2 represents the results obtained by our algorithm with a particularly simple selection method $\hat{f}_\epsilon = (\hat{f}_\epsilon^{\,i_0} + \hat{f}_\epsilon^{\,i_1})/2$. We can observe from Figure 1 that our algorithm, while having similar complexity, improves the accuracy compared to the Schmidl-Cox algorithm by 7dB at 0dB SNR, and then by 9dB from 3dB SNR on. We can also observe that our algorithm is around 4.5dB inferior to the algorithm by Morelli and Mengali at 0dB SNR, and around 1dB inferior at high signal-to-noise ratios. In Figure 2, we compare our algorithm with the algorithm by Morelli and Mengali taking into account quantization of arg (the argument of a complex number) function that takes place in all three algorithms studied in this paper. We can observe that under quantization, which is inevitable in practice, our algorithm performs much better than the algorithm by Morelli and Mengali.

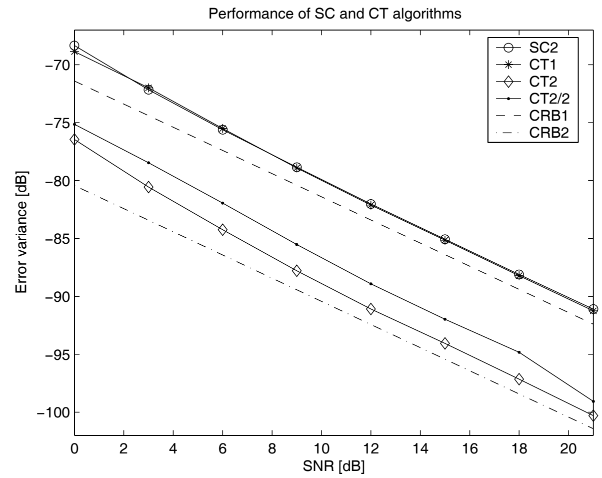

Fig. 1. *Comparison of frequency synchronization algorithms. SC2 - the algorithm by Schmidl and Cox with two training sequences. CT1 and CT2 - the algorithm proposed here with one and two training sequences, respectively. CT2/2 line - the variant of the algorithm proposed here where* $\hat{f}_\epsilon = \mathrm{mean}(\hat{f}_\epsilon^{\,i_0}, \hat{f}_\epsilon^{\,i_1})$. *CRB1 and CRB2 - Cramer-Rao bounds for estimation using one and two training sequences, respectively. The algorithm by Morelli and Mengali operates very close to CRB2 curve.*

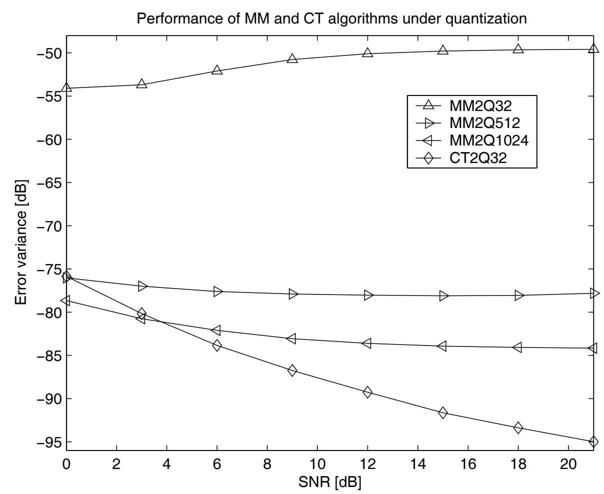

Fig. 2. *Comparison between our algorithm and the algorithm by Morelli and Mengali under quantization. CT2Q32 - our algorithm with two training sequences and 32-level quantization of* arg *function. MM2QX curves - Morelli and Mengali algorithm with 2 training sequences and X-level quantization.*

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