# RECONSTRUCTION FROM MISSING FUNCTION AND DERIVATIVE SAMPLES AND OVERSAMPLED FILTER BANKS

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**ABSTRACT**

This paper deals with two-channel sampling using an oversampled filter bank. It emphasizes sampling of one function $f$ and its derivative $f'$, at a rate higher than the critical minimum rate (oversampling), and the problem of reconstructing the function even when a finite number of samples (of $f$ or $f'$) are unknown. The motivation for considering this problem is the need to add redundancy to data, as in conventional channel coding, and the convenience of doing it in a domain suited to the data, as in joint source-channel coding.

## 1. INTRODUCTION

Real time multimedia transmission over IP networks is prone to packet loss. One approach to circumvent these losses without retransmissions and further delays is the reconstruction of the lost data from the received packets. This can be done using conventional error control coding or using techniques that are roughly equivalent to joint source-channel DFT codes [1, 2, 3]. An introduction to DFT codes, reconstruction and frames can be found in [4].

In addition to DFT codes, which deal with vectors of $N$ samples and can be thought of as block codes over the complex field, oversampled sampling series may also be used [5]. Implementations are described in [6, 7]. In the context of DFT codes, the potential of two-channel DFT codes has been noted [8]. So far, however, two-channel oversampled filter banks and the associated sampling series have not been studied in this context.

Multichannel sampling is a well-known topic [9, 10]. In the two-channel case, the given data are samples of the convolution of the function $f$ with the analysis filters $h_0$ and $h_1$, as shown in Fig. 1. The problem is that of reconstructing $f$ from these samples. It turns out that finite-energy signals can be reconstructed when the analysis filters are sampled at half the Nyquist rate. It results that the sampling rate, in the overall system, is exactly the Nyquist rate: the sampling frequency in each channel is equal to the maximum frequency of $f$.

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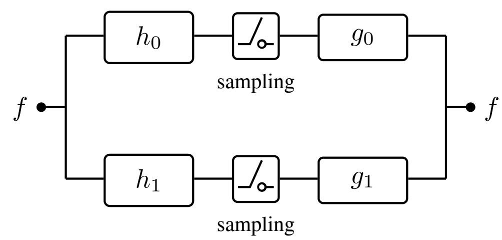


**Fig. 1**. Two-channel sampling as a filter-bank: $f$ is convolved with the analysis filter banks, $h_0$ and $h_1$, and the obtained signals are sampled. The synthesis filters $g_0$ and $g_1$ reconstruct $f$ from the two sets of samples.

Function and derivative sampling is a particular case of two-channel sampling, where one of the analysis filters returns samples of $f$ and the other returns samples of $f'$. As in the two-channel case, the reconstruction of a finite-energy function $f$ bandlimited to $\omega_a$ Hz from samples of itself and its derivative is possible when each of these are sampled at half the Nyquist rate, that is, $\omega_a$ Hz, and the reconstruction formula [10] is given by

$$f(t) = \sum_{k=-\infty}^{\infty} \left\{ f\left(\frac{k}{\omega_a}\right) \operatorname{sinc}^2\left[\omega_a\left(t - \frac{k}{\omega_a}\right)\right] + f'\left(\frac{k}{\omega_a}\right)\left(t - \frac{k}{\omega_a}\right) \operatorname{sinc}^2\left[\omega_a\left(t - \frac{k}{\omega_a}\right)\right]\right\}, \quad (1)$$

where, as usual, $\operatorname{sinc}(x) = \sin(\pi x)/(\pi x)$.

In the classical sampling theorem, the signal is reconstructed from one set of samples. When sampling at the critical (minimum) rate, the loss of even one sample prevents reconstruction. However, when sampling at a rate higher than the critical rate (oversampling), exact recovery remains possible even when a finite number of samples is lost [11, 12, 13], a fact that remains true even for Kramer sampling series [11].

For two-channel sampling, and in particular for function and derivative sampling, when sampling at half the critical rate in each channel, the loss of one or more samples (of $f$ or $f'$) also yields impossibility of reconstruction, in general. The purpose of this paper is to consider sampling at a rate higher than the critical rate (oversampling), and the problem of reconstructing from lost samples in this case.

Let $\omega_s$ denote the sampling frequency in each channel, and $\omega_a$ the maximum frequency of the finite-energy signal $f$. In the oversampling case, $\omega_a = r\omega_s$, where $0 < r < 1$.

The case $0 < r \leq 1/2$ is trivial, since it implies oversampling in each channel separately, yielding possible reconstruction from lost samples by the existing theory (treating any of the two channels independently of the other).

When $1/2 < r < 1$, $\omega_s$ takes values in the interval $(\omega_a, 2\omega_a)$ and each channel is undersampled. However, because the overall sampling frequency is $2\omega_s$, the filter bank as a whole is oversampled. This is the only interesting case. We will show that in this case the reconstruction from lost samples is indeed possible.

## 2. OVERSAMPLING

The general sampling formula for two-channel sampling can be adapted to the oversampling case by projecting (filtering) on the subspace of signals band-limited to $\omega_a$ Hz.

Assuming for simplicity and without loss of generality that $\omega_s = 1$ Hz and, thus, $\omega_a = r$, the reconstruction formula would be

$$f(t) = \sum_{k=-\infty}^{\infty} a(k)s_1(t-k) + b(k)s_2(t-k), \qquad (2)$$

where $a(k)$ and $b(k)$ are the sampled outputs of the analysis filters, and $s_1$ and $s_2$ are the projections of their impulse responses on the subspace of signals band-limited to $r$ Hz, $1/2 < r < 1$.

In the case of function and derivative sampling, it can be verified that the reconstruction formula obtained by low-pass filtering the signal given in (1) is

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\omega_s}\right)\left\{2r(1-r)\text{sinc}\left[2r\omega_s\left(t-\frac{k}{\omega_s}\right)\right] + r^2\text{sinc}^2\left[r\omega_s\left(t-\frac{k}{\omega_s}\right)\right]\right\} + f'\left(\frac{k}{\omega_s}\right) r^2\left(t-\frac{k}{\omega_s}\right)\text{sinc}^2\left[r\omega_s\left(t-\frac{k}{\omega_s}\right)\right].$$

Assuming, as above, that $\omega_s = 1$ and, thus, $\omega_a = r$, this reduces to

$$f(t) = \sum_{k=-\infty}^{\infty} f(k)s_1(t-k) + f'(k)s_2(t-k), \qquad (3)$$

where

$$s_1(t) = 2r(1-r)\text{sinc}[2r(t)] + r^2\text{sinc}^2[r(t)]$$

and

$$s_2(t) = r^2 t\text{sinc}^2[r(t)].$$

## 3. RECONSTRUCTION FROM MISSING SAMPLES

For simplicity, we will deal with the function and derivative case, and the modifications necessary to check other oversampled two-channel structures will be inferred from these.

Consider $i_1, ..., i_n$ to be the positions of the missing samples in $\{f(k)\}_k$ and/or $\{f'(k)\}_k$. Then, (3) can be presented in terms of the unknown positions,

$$f(i_m) = \sum_{k=1}^{n} f(i_k)s_1(i_m - i_k) + f'(i_m)s_2(i_m - i_k) + h(i_m) \qquad (4)$$

where

$$h(i_m) = \sum_{k\neq i_1,\ldots,i_n} f(k)s_1(i_m-k) + f'(k)s_2(i_m-k),$$

for $m = 1, .., n$. In matrix form, (4) can be expressed as

$$f = S_1 f + S_2 f' + h, \qquad (5)$$

where $(S_1)_{mk} = s_1(i_m - i_k)$, $(S_2)_{mk} = s_2(i_m - i_k)$ and $h = \begin{bmatrix} h(i_1) & \ldots & h(i_n)\end{bmatrix}^T$.

### 3.1. Missing Function Samples

When the samples $\{f(i_k)\}_{k=1}^n$ are missing, but $\{f'(i_k)\}_{k=1}^n$ are known, then recovering $f$ can be obtained from (5) as

$$(I - S_1)f = S_2 f' + h$$

as long as $I - S_1$ is invertible. If $I - S_1$ is indeed invertible, then 1 cannot be an eigenvalue of $S_1$. This yields that the quadratic form of $S_1$ satisfies the condition $x^H S_1 x \neq \|x\|^2$, for $x \neq 0$.

To see if this is true, for $x \neq 0$, one obtains that

$$\begin{aligned} x^H S_1 x &= \sum_m \sum_k x_m^* x_k s_1(i_m - i_k) \\ &= \sum_m \sum_k x_m^* x_k \int_{-r}^{r} \hat{s}_1(\omega) e^{i2\pi\omega(i_m - i_k)} d\omega \\ &= \int_{-r}^{r} \hat{s}_1(\omega) \sum_m \sum_k x_m^* x_k e^{i2\pi\omega(i_m - i_k)} d\omega \\ &= \int_{-r}^{r} \hat{s}_1(\omega) \underbrace{\left| \sum_k x_k e^{i2\pi\omega i_k} \right|^2}_{M(\omega)} d\omega. \end{aligned} \quad (6)$$

The function $M$ is a periodic function with period of 1 Hz. It then follows,

$$\begin{aligned} x^H S_1 x &= \int_{-r}^{-1/2} \hat{s}_1(\omega) M(\omega) d\omega + \\ &\quad \int_{-1/2}^{1/2} \hat{s}_1(\omega) M(\omega) d\omega + \\ &\quad \int_{1/2}^{r} \hat{s}_1(\omega) M(\omega) d\omega \\ &= \int_{1-r}^{1/2} \hat{s}_1(\omega - 1) M(\omega - 1) d\omega + \\ &\quad \int_{-1/2}^{1/2} \hat{s}_1(\omega) M(\omega) d\omega + \\ &\quad \int_{-1/2}^{r-1} \hat{s}_1(\omega + 1) M(\omega + 1) d\omega \\ &= \int_{-1/2}^{1/2} \left[ \hat{s}_1(\omega - 1) \chi_{[1-r,1/2]} + \hat{s}_1(\omega) \right. \\ &\quad \left. + \hat{s}_1(\omega + 1) \chi_{[-1/2,r-1]} \right] M(\omega) d\omega. \end{aligned}$$

Due to the fact that $1/2 < r < 1$, there are at most two overlaps. In this manner, $\hat{s}_1(\omega - 1)$ never overlaps with $\hat{s}_1(\omega + 1)$. This also holds true for two-channel oversampling, for any of the two channels.

For function and derivative sampling, the Fourier transform of $s_1$ is

$$\hat{s}_1(\omega) = 1 - |\omega|, \quad \omega \in [-r, r]. \quad (7)$$

Hence, when the overlapping does occur either

$$\hat{s}_1(\omega) + \hat{s}_1(\omega - 1) = 1$$

is satisfied, or

$$\hat{s}_1(\omega) + \hat{s}_1(\omega + 1) = 1.$$

Nevertheless, there is a neighborhood around zero where overlapping is absent. Because of this,

$$x^H S_1 x < \int_{-1/2}^{1/2} M(\omega) d\omega = \|x\|^2.$$

Consequently, $I - S_1$ is invertible and, therefore, reconstruction from missing samples of $\{f(k)\}_k$ is possible.

The more general two-channel case can be treated similarly. Given the Fourier transform of each analysis filter, $\hat{s}_1(\omega)$ and $\hat{s}_2(\omega)$, if

$$\hat{s}_1(\omega) + \hat{s}_1(\omega - 1) + \hat{s}_1(\omega + 1) \leq 1$$

for all $\omega$ belonging to $I = (-1/2, 1/2)$, and if the inequality is strict over a subinterval of $I$, reconstruction is possible.

### 3.2. Missing Derivative Samples

In turn, when the samples $\{f(i_k)\}_{k=1}^n$ are known and the missing ones are $\{f'(i_k)\}_{k=1}^n$, then (5) becomes

$$S_2 f' = (I - S_1) f - h.$$

However, $S_2$ is antisymmetric. For odd order, it can be verified that $S_2$ is singular. Then, this equation is not satisfactory for the recovery of $\{f'(i_k)\}_{k=1}^n$. Nevertheless, the derivation of (5) yields

$$f' = S_1' f + S_2' f' + h'.$$

Then, the recovery of $\{f'(i_k)\}_{k=1}^n$ will be held by

$$(I - S_2') f' = S_1' f + h'$$

if $I - S_2'$ is invertible. The process to study the invertibility of $I - S_2'$ is analogous to that of $I - S_1$.

For $x \neq 0$,

$$\begin{aligned} x^H S_2' x &= \sum_m \sum_k s_2'(i_m - i_k) \\ &= \int_{-r}^{r} \widehat{s_2'} \underbrace{\left| \sum_k x_k e^{i2\pi\omega i_k} \right|^2}_{M(\omega)} d\omega. \end{aligned}$$

Recall that $M$ is a periodic function with period of 1 Hz and note that

$$\begin{aligned} \widehat{s_2'} &= i\omega \hat{s_2} \\ &= i\omega(-i\text{sgn}(\omega)) \\ &= |\omega|. \end{aligned}$$

Then, similarly to $S_1$,

$$\begin{aligned} x^H S_2' x &= \int_{-1/2}^{1/2} \left[ |\omega - 1| \chi_{[1-r,-1/2]} + |\omega| \right. \\ &\quad \left. + |\omega + 1| \chi_{[-1/2,r-1]} \right] M(\omega) d\omega. \end{aligned}$$

Since $1/2 < r < 1$, there are at most two overlaps: $\widehat{s_2'}(\omega)$ either sums with $\widehat{s_2'}(\omega - 1)$ or with $\widehat{s_2'}(\omega + 1)$, but never with both simultaneously. Moreover, when the sums do occur, then

$$|\omega| + |\omega - 1|\chi_{[-1/2,r+1]} = 1$$

or

$$|\omega| + |\omega - 1|\chi_{[1-r,1/2]} = 1.$$

However, around zero there is a neighborhood where overlapping does not occur. It then follows,

$$x^H S_2' x < \int_{-1/2}^{1/2} M(\omega) d\omega = \|x\|^2.$$

So, analogously to $S_1$, 1 is not an eigenvalue of $S_2'$ and, hence, $I - S_2'$ is invertible. Therefore, the recovery of missing samples of $f'$ is also possible.

It is interesting to observe, however, that the equations to recover the missing samples of $f$ and of $f'$ are not the same, although related.

## 4. CONCLUSION AND FUTURE WORK

Joint source-channel DFT codes have been studied and applied by several authors. A recent work addressed a two-channel DFT code, claiming that the two-channel structure is numerically stable for bursty losses [8]. This work addressed the two-channel filter bank (equivalently, the two-channel sampling theorem) and established that recovery from missing samples in one of the two channels is possible provided the system as a whole is oversampled (that is, each of the two channels is sampled slightly above the maximum frequency of the input signal). In any case, the reconstruction algorithm is simple and efficient: the reconstruction of $N$ lost samples in one of the two channels can be performed by solving a set of $N$ linear equations. Although the results given can be used to derive bounds for the eigenvalues of the matrices involved, the numerical stability of the problem deserves further study. Comparisons with the single-channel case [12], for example, would be interesting.

## 5. REFERENCES


[1] J. M. N. Vieira and P. J. S. G. Ferreira, "Interpolation, spectrum analysis, error-control coding, and fault-tolerant computing," in *Proceedings of the IEEE International Conference on Acoustics, Speech, and Signal Processing, ICASSP 97*, Munich, Germany, Apr. 1997, vol. III, pp. 1831–1834.

[2] P. J. S. G. Ferreira, "Stability issues in error control coding in the complex field, interpolation, and frame bounds," *IEEE Sig. Proc. Letters*, vol. 7, no. 3, pp. 57–59, Mar. 2000.

[3] G. Rath and C. Guillemot, "Performance analysis and recursive syndrome decoding of DFT codes for bursty erasure recovery," *IEEE Trans. Signal Processing*, vol. 51, no. 5, pp. 1335–1350, May 2003.

[4] P. J. S. G. Ferreira, "Mathematics for multimedia signal processing II — discrete finite frames and signal reconstruction," in *Signal Processing for Multimedia*, J. S. Byrnes, Ed., pp. 35–54. IOS Press, 1999.

[5] F. A. Marvasti and M. Nafie, "Sampling theorem: A unified outlook on information theory, block and convolutional codes," *IEICE Trans. Fundam. Electron. Commun. Comput. Sci.*, vol. E76–A, no. 9, pp. 1383–1391, Sept. 1993.

[6] M. Nafie and F. M. Marvasti, "Implementation of recovery of speech with missing samples on a DSP chip," *Electron. Letters*, vol. 30, no. 1, pp. 12–13, Jan. 1994.

[7] C. K. W. Wong, F. Marvasti, and W. G. Chambers, "Implementation of recovery of speech with impulsive noise on a DSP chip," *Electron. Letters*, vol. 31, no. 17, pp. 1412–1413, Aug. 1995.

[8] P. J. S. G. Ferreira and J. M. N. Vieira, "Stable DFT codes and frames," *IEEE Sig. Proc. Letters*, vol. 10, no. 2, pp. 50–53, Feb. 2003.

[9] J. L. Brown, Jr., "Sampling of bandlimited signals: Fundamental results and some extensions," in *Handbook of Statistics*, N. K. Bose and C. R. Rao, Eds., vol. 10, pp. 59–101. Elsevier Science, Amsterdam, 1993.

[10] J. R. Higgins, *Sampling Theory in Fourier and Signal Analysis: Foundations*, Oxford University Press, Oxford, 1996.

[11] P. J. S. G. Ferreira, "Incomplete sampling series and the recovery of missing samples from oversampled band-limited signals," *IEEE Trans. Signal Processing*, vol. 40, no. 1, pp. 225–227, Jan. 1992.

[12] P. J. S. G. Ferreira, "The stability of a procedure for the recovery of lost samples in band-limited signals," *Sig. Proc.*, vol. 40, no. 3, pp. 195–205, Dec. 1994.

[13] F. Marvasti, Ed., *Nonuniform Sampling: Theory and Practice*, Kluwer Academic / Plenum Publishers, New York, 2001.