# A HYBRID WAVELET AND RIDGELET APPROACH FOR EFFICIENT EDGE REPRESENTATION IN NATURAL IMAGES

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## ABSTRACT

Wavelets, fundamental component of the latest image coding standard JPEG 2000, have proven to be a very useful tool in representation and coding of 2D images. They are also a powerful instrument in catching zero-dimensional singularities, though less efficient in dealing with 'edge' singularities. Ridgelets, by definition, are a powerful tool in catching and representing line singularities in bidimensional space.

In this paper we propose a hybrid approach that combines both Ridgelets and Wavelets for a more efficient representation of 2-D images with edges.

Results confirm the potential of the combined use of Wavelets and Ridgelets as efficient representation, showing substantial improvements when compared to only Wavelets.

## 1. INTRODUCTION

Images are generally described via orthogonal, non-redundant transforms like wavelet or discrete cosine transform. Their main limitation is that these bases do not take advantage of the regularity of many signal structures. Indeed, these basis are composed of vectors having a support which is not adapted to the elongation of the signal structures, such as regular edges. In this framework sparse representation of edges in natural images constitutes a challenging problem. Wavelet transforms (DWT), powerful instrument in catching zero-dimensional singularities, are less successful in dealing with 'edge' singularities.

The Ridgelet transform offers the possibility to achieve a very compact representation of linear singularities in images [1, 2]. Instrumental in the implementation of the Ridgelet is the Radon transform, a powerful tool to extract lines in edge dominated images.

In this paper we introduce a hybrid scheme that combines Wavelets and Ridgelets, aiming at an efficient representation for images with edges. In the following we illustrate some of the properties and limitations of the Ridgelet and Radon transforms and we propose an implementation that overcome these limitations and that can be integrated into a DWT representation scheme.

## 2. THE PROBLEM OF EDGE REPRESENTATION

Let us assume to have a 2D function $g$ defined on $[0,1]^2$. Let us consider that $g$ is smooth away from a discontinuity along a $C^2$ curve $\gamma$. A grid of squares of $2^{-j}$ by $2^{-j}$ has order $2^j$ intersecting $\gamma$. At level $j$ of the two-dimensional wavelet pyramid, each wavelet is localized near a corresponding square of side $2^{-j}$ by $2^{-j}$. There are therefore $O(2^j)$ wavelets which 'feel' the discontinuity along $\gamma$.

## 3. THE RIDGELET TRANSFORM

### 3.1. The Continuous Ridgelet Transform

We start by briefly reviewing the continuous ridgelet transform, defined by Candès and Donoho in [1], stemming from the Radon transform, instrumental in its implementation. Given an integrable bivariate function $f(x_1,x_2)$, its Radon transform (RDN) is defined by:

$$RDN_f(\theta,t) = \int_{R^2} f(x_1,x_2)\delta(x_1\cos\theta + x_2\sin\theta - t)dx_1dx_2 \cdot \quad (1)$$

Basically the Radon operator maps the spatial domain into the projection domain ($\theta$, $t$), in which each point corresponds to a straight line in the spatial domain; conversely, each point in the spatial domain becomes a sine curve in the projection domain.

The Continuous Ridgelet Transform (CRT) is simply the application of a mono-dimensional wavelet ($\psi_{a,b}(t) = a^{-1/2}\psi((t-b)/a)$) to the slices of the Radon transform:

$$CRT(a,b,\theta) = \int_R \psi_{a,b}(t)RDN_f(\theta,t)dt =$$
$$= \int_{R^2} \psi_{a,b,\theta}(x_1,x_2)f(x_1,x_2)dx_1dx_2\,, \quad (2)$$

where the ridgelets $\psi_{a,b,\theta}(\bar{x})$ in 2-D are defined from a wavelet-type function $\psi(t)$ as:

$$\psi_{a,b,\theta}(x_1,x_2) = a^{-1/2}\psi((x_1\cos\theta + x_2\sin\theta - b)/a)\,. \quad (3)$$

This shows that the ridgelet function is constant along the lines where $x_1 \cos\theta + x_2 \sin\theta = const.$

Comparing ridgelets with wavelets we observe that the parameters of the former are scale factor and line position (respectively $a$ and $(b,\theta)$ in (2)), while the latter uses scale factor and point position. As a consequence, wavelets are very effective in representing objects with isolated point singularities, while ridgelets are very effective in representing objects with singularities along straight lines.

### 3.2. The Finite Ridgelet Transform and the wrap around problem of the Finite Radon Transform

For practical applications, the development of a discrete version of the ridgelet transform and its algorithmic implementation is a challenging problem. Beacause of the radial nature of the Ridgelets, implementation based on the direct discretization of the continuous formula need interpolation in polar coordinates and this causes redundancy or non perfect reconstruction. In [3, 4] authors take a redundant approach and though it has the great merit of providing invertibility, it introduces a factor four in oversampling, that makes that not suitable for compression or efficient representation.

In [5, 6], Do and Vetterli propose a new procedure that results to be invertible, orthogonal and achieves perfect reconstruction: the *Finite RIdgelet Transform* (FRIT). FRIT is based on the *Finite RAdon Transform* (FRAT), which is defined as summations of image pixels over a certain set of "lines". Those lines are defined in a finite geometry in a similar way as the lines for the continuous Radon transform in the Euclidean geometry. Denote $Z_p = \{0,1,..,p-1\}$, where $p$ is a prime number. Note that $Z_p$ is a finite field with modulo $p$ operations. The FRAT of a real discrete function $f$ on the finite grid $Z_p^2$ is defined as:

$$FRAT_f(k,l) = \frac{1}{\sqrt{p}} \sum_{(i,j)\in L_{k,l}} f(i,j). \qquad (4)$$

Here $L_{k,l}$ denotes the set of points that make up a line on the lattice $Z_p^2$, i.e.

$$L_{k,l} = \begin{cases} \{(i,j) : j = (ki+l)(\mathrm{mod}\, p), i \in Z_p\} \; if\ 0 \le k < p \\ \{(l,j) : j \in Z_p\} \; if\ k = p \end{cases} . \qquad (5)$$

The inverse transform is obtained through the *Finite Back-Projection operator* (FBP) defined as a sum of Radon coefficients of all the lines that go through a given point. Here $f$ is supposed to be a zero-mean image:

$$FBP_r(i,j) = \frac{1}{\sqrt{p}} \sum_{(k,l)\in P_{i,j}} FRAT_f(k,l). \qquad (6)$$

From (5) it can be found that $P_{i,j}$ is:

$P_{i,j} = \{(k,l) : l = (j-ki)(\mathrm{mod}\ p), k \in Z_p\} \cup \{(p,i)\}$.

Substituting (6) into (4) we obtain that

$$FBP_r(i,j) = \frac{1}{p} \sum_{(k,l)\in P_{i,j}} \sum_{(i',j')\in L_{k,l}} f(i',j') =$$

$$= \frac{1}{p}\left( \sum_{(i',j')\in Z_p^2} f(i',j') + p \cdot f(i,j) \right) = f(i,j) \qquad (7)$$

and so the perfect reconstruction is achieved.

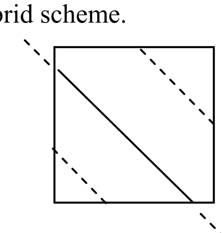
FRIT needs an input image of size $p$x$p$, where $p$ is a prime number, and this is an important limitation of this algorithm. Moreover wavelets usually require a dyadic length signal and this is absolutely incompatible with the FRAT output (that is a matrix $p$x($p$+1)). In the test we made, we extended the length of the signal from $p$ to $m$, where $m$ is defined as:

$m = \min\{n \in N : (n > p)\ \text{and}\ (n = 2^d), d \in N\}$.

In our experiments we took $p$=31 and so $m$=32.

The just introduced FRIT has again the great merit of invertibility and orthogonality and it is not redundant, which makes it more suitable for efficient representation and compression application, though it has the problem of the wrap-around that introduces quite annoying artifacts when looking for sparse representation.

We remind to [6] for a more detailed explanation of the wrap around problem; because of space limitation here we just remind that the behavior of the FRAT is such that the generic coefficient FRAT(k,l) covers a line and its parallel in the $Z_p$ space (see Fig. 1). This kind of problem exists for all the directions but the horizontal and vertical; for them no wrap around is present, therefore these are suitable directions for efficient edge representation in a hybrid scheme.



*FRAT(k,l)*

Fig. 1 The wrap around effect of the FRAT

## 4. THE HYBRID APPROACH

We have introduced the problem of Wavelets on edges and proposed the Ridgelets as a solution for that. At the same time, when talking about the digital implementation of the Ridgelet we have underlined the limitations of the FRIT and the wrap around problem. We also stressed that there is no wrap around in the horizontal and vertical direction.

In this section we introduce a scheme that combines the Wavelets and the Ridgelets. In particular we will use the Ridgelets for the representation of the information present in the $HL_1$ and $LH_1$ subbands of the wavelet decomposition. In these subbands is in fact present the high frequency information in the horizontal and vertical direction, therefore horizontal and vertical edges.

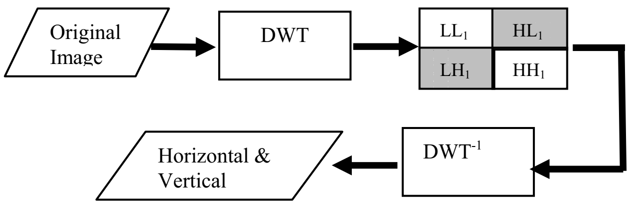



Fig. 2 Extraction of edges using the $HL_1$ and $LH_1$ subbands

Once extracted the edges then they are partitioned in blocks 32x32. The partitioning provides a solution to the 'stretching effect' of the Digital Ridgelet Transform (DRT); namely, by doing the partitioning we reduce the risk of finding artificial edges that might span throughout all the image.

The block diagram of the proposed approach is presented in Fig 3. At first we perform an edge approximation, retaining N coefficients of the Wavelet Transform; the resulting mean square error represents the reference starting point as far as the distortion is concerned. Then we compute a Ridgelet analysis of the first block of the edges partitioning, selected with an energy criterion or simply in a raster scan order. At each step we retain $n$ ($n$=1 at the first step) Ridgelet coefficients. Once obtained the Ridgelet reconstruction of the first block we subtract it from the edges in the image domain and we make a Wavelet analysis of the resulting residual image retaining N-n coefficients. Finally we add the Wavelet residual reconstruction with the Ridgelet edges reconstruction, obtaining in this way a hybrid reconstruction. Now, if the mean square error between the edges and our hybrid reconstruction is smaller than the one between the edges and the Wavelet reconstruction then we add one Ridgelet coefficient for the Ridgelet analysis of the block and restart the procedure (point B in Fig. 3), otherwise we compute the following block (point A in Fig. 3) and so on. The aim is to catch the energy along the edges with only a few Ridgelet coefficients obtaining, step by step, a residual image with less and less edges and so containing only parts suitable to be analyzed with the DWT.

At every step we then check if the reconstruction is improving or not, and if not then we halt the DRT analysis. This procedure is executed for all the blocks.

The feedback control provides a novel way to properly employ the DRT, limiting its use to the circumstances

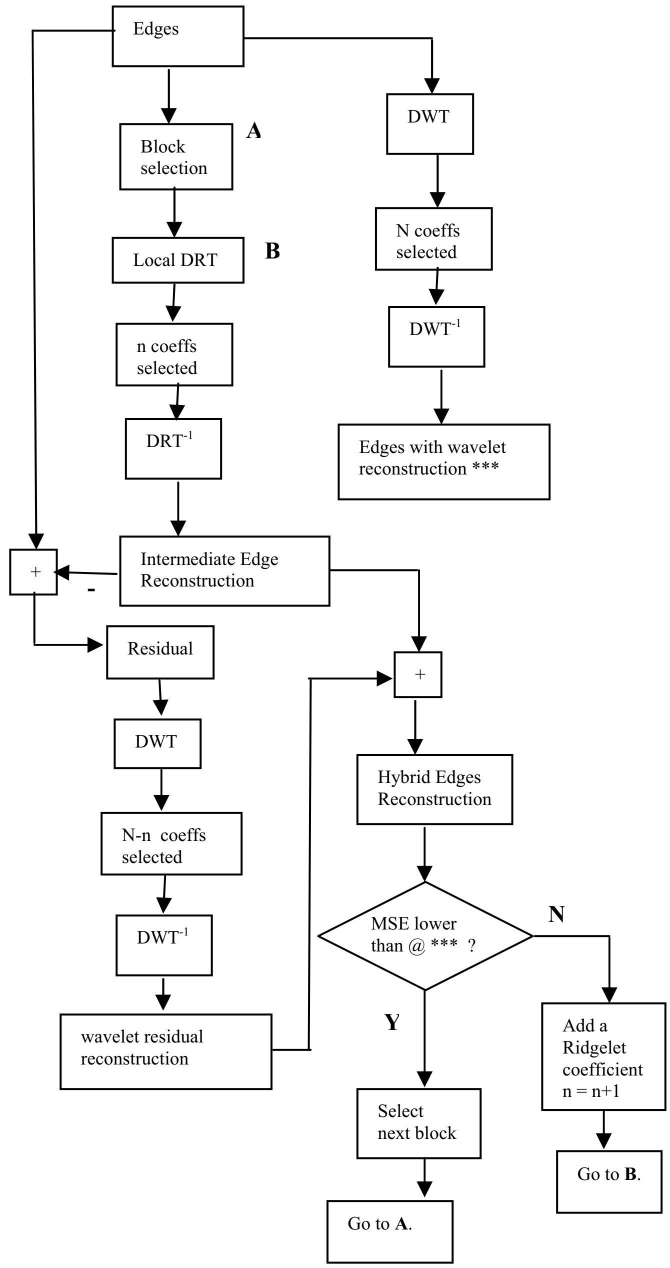



Fig. 3 Block diagram of the proposed approach

where a real advantage is provided. In this hybrid approach each of the two orthogonal transform DWT and DRT does provide in the end the best representation for those patterns of the image for which it is best suited.

Elongated structures are represented with few DRT coefficients.

## 5. RESULTS

The results are reported hereafter in Fig. 4.a and Fig. 4.b as rate distortion curves, with MSE versus Total number of coefficients needed. The blue curve is obtained using

DWT coefficients only, while the red curve is obtained using the proposed hybrid approach (DRT+DWT).
The ones reported here are typical results obtained for the image of the well known sequences Hall and Stefan.
It is possible to notice that, on average, the proposed approach achieves the same distortion with 20-30% less coefficients, with peaks of 40% for Stefan. In Fig 5 we show the edges of the sequence 'Hall' extracted from the $HL_1$ and $LH_1$ subbands and the reconstruction with 1000 hybrid coefficients and 1000 DWT coefficients.

## 6. CONCLUSIONS

In this paper we proposed a hybrid approach that combines both Ridgelets and Wavelets for a more efficient representation of 2-D images with edges.

Results confirm the potential of the combined use of Wavelets and Ridgelets showing substantial improvements when compared to Wavelets. Further investigation will be done in the direction of finalizing the coding scheme adopting an arithmetic encoder.

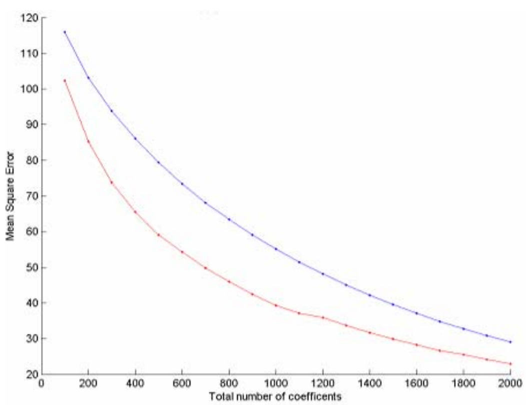



Fig. 4.a Rate distortion curves (MSE versus number of coefficient) of the Wavelets (blue) versus the proposed approach for the frame Hall

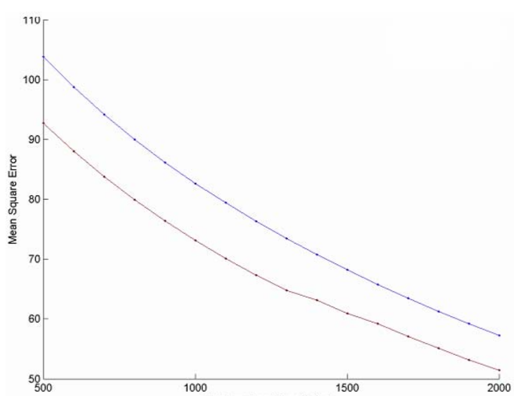



Fig. 4.b Rate distortion curves (MSE versus number of coefficient) of the Wavelets (blue) versus the proposed approach for the frame Stefan

## 7. REFERENCES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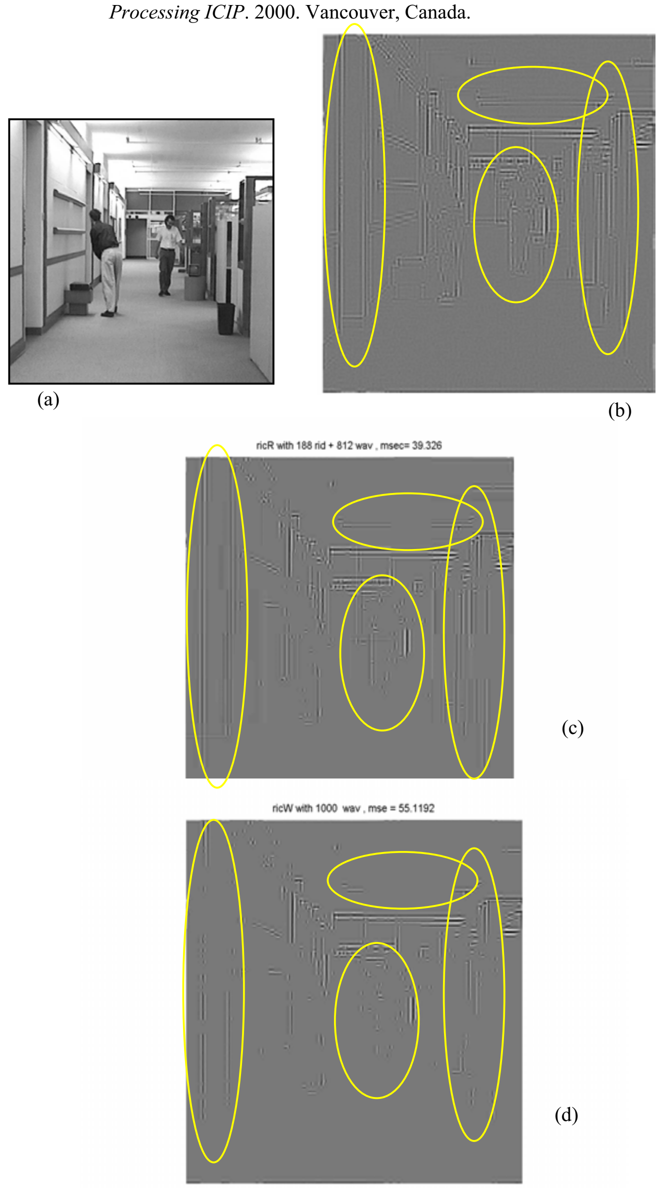


Fig. 5 Sequence 'Hall' (a): the Edges extracted from the $HL_1$ and $LH_1$ subbands (b) and the reconstruction with 1000 hybrid coefficients (b) and 1000 DWT coefficients (d).