OPTIMIZATION OF ORTHOGONAL WAVELETS FOR IMAGE COMPRESSION

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ABSTRACT

A technique is developed where a wavelet is matched to a 1-D or 2-D signal, with the goal of increasing coding gain. This is done by maximizing the projection of the signal onto the scaling subspace. For 1-D signals, the results show that the scaling subspace (low pass) signal closely matches (subjectively and objectively) the original signal. For 2-D signals, the optimized wavelets give superior coding results when compared with the Daubechies wavelets. For low compression ratios the PSNR gain is over 1.5 dB. This gain drops off at higher compression ratios, possibly due to the lower number of vanishing moments in the optimized wavelets.

1. INTRODUCTION

Over the last couple of years, the wavelet transform [1] has been receiving increased attention from the signal processing community, especially in the field of image compression [2]. State-of-the-art coding algorithms have been implemented to exploit the redundancies inherent in the transform coefficients to produce excellent results in image compression, e.g. the EZW algorithm [3].

The 1-D wavelet transform is usually implemented by a filter bank, as shown in figure 1. There have been a vast number of pre-designed wavelet and scaling functions to be found in the literature. These functions are independent of what signal is to be analyzed. Work has been done to develop methods for matching a signal to a wavelet [4–6]. However all these methods are only used for the 1-D case, and not specifically for application in signal compression.

This paper describes a method to match the wavelet to both a 1-D and 2-D signal (e.g. an image) with the aim of increased image compression performance. We will also study the properties of the new basis functions matched to these signals.

2. MULTIRESOLUTION DECOMPOSITION AND THE DISCRETE WAVELET TRANSFORM

In the context of multiresolution analysis, $\mathcal{V}_j$ is defined as a subspace of $L^2(\mathbb{R})$ which spans the scaling function $\varphi(2^j t - k)$ for all $k \in \mathbb{Z}$. $\mathcal{W}_j$ is defined as the orthogonal compliment of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$.

In figure 1, $x(n)$ represents the highest possible resolution version of a continuous time signal $x(t) \in L^2(\mathbb{R})$. Hence, $x(n) = cA_0(n) \in \mathcal{V}_0$ and

$$x(t) = \sum_k x(k)\, \varphi(t-k) \tag{1}$$

The transform coefficients, $cA_{-1}(n)$ and $cD_{-1}(n)$, represent the signal $x(n)$ at scale $j = -1$. In other words, they represent the projection of $x(t)$ on the scaling subspace, $\mathcal{V}_{-1}$, and the wavelet subspace, $\mathcal{W}_{-1}$, respectively.

To obtain an even lower resolution version of $x(n)$ as cA_{-2} and cD_{-2}, the signal cA_{-1} is passed through the same filter bank in place of $x(n)$. This leads to the following subspace relationship for the DWT:

$$L^2(\mathbb{R}) = \mathcal{V}_0 \oplus \mathcal{W}_{-1} \oplus \mathcal{W}_{-2} \oplus \mathcal{W}_{-3} \oplus \cdots \tag{2}$$

2.1. Necessary Conditions for the Orthogonal DWT

For the multiresolution framework described above to hold, certain conditions need to be imposed on the orthogonality of both the scaling function ($\varphi(t)$), and the wavelet ($\psi(t)$). These orthogonality conditions can be found in [7]. Using the basic recursive equations for $\varphi(t)$ and $\psi(t)$, the orthogonality relationships can be written as

$$\sum_k h_0(k)\, h_0(k-2n) = \delta(n) \tag{3}$$

$$\sum_k h_1(k)\, h_1(k-2n) = \delta(n) \tag{4}$$

$$\sum_k h_0(k)\, h_1(k-2n) = 0 \tag{5}$$

Also from the basic recursive equations, the following can be derived [7]:

$$\sum_k h_0(k) = \sqrt{2} \tag{6}$$

$$\sum_k h_1(k) = 0 \tag{7}$$

3. OPTIMIZATION FOR A 1-D SIGNAL

In [6], a method was proposed to match a wavelet to a 1-D signal. This method had the following errors/limitations:

1. No constraints were placed on the solution to satisfy the necessary conditions in section 2.1.
2. To avoid a trivial solution to the problem, one of the filter coefficients of h_1 was set to 1.

This section revisits and corrects the procedure of [6] for 1-D systems, and further develops the technique for 2-D systems in section 4.

In order to match a wavelet to a 1-D signal, its projection onto the $\mathcal{W}_j$ subspace should be minimized, which in turn would give

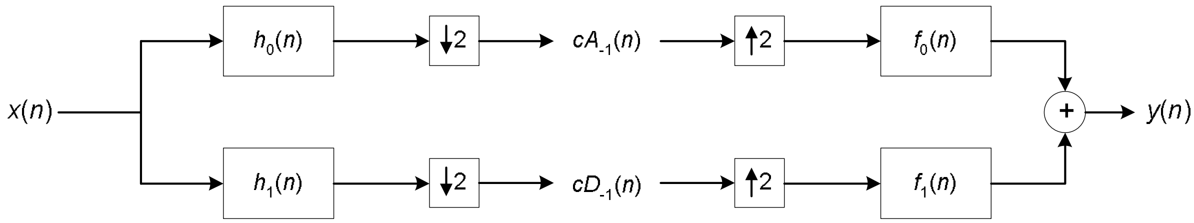


Fig. 1. Implementation of a single stage wavelet transform using a filter bank.

a maximum projection in $\mathcal{V}_j$. This is desirable for signal compression, where the coefficients in $\mathcal{W}_j$ are usually discarded or coarsely quantized.

Now, in a perfect reconstruction (PR) filter bank, $y(n) = x(n)$ in figure 1. This PR filter bank can be realized by setting [1]

$$h_1(n) = (-1)^{(n+1)} h_0(N-n-1) \tag{8}$$
$$f_0(n) = h_0(N-n-1) \tag{9}$$
$$f_1(n) = h_1(N-n-1) \tag{10}$$

where N is the length of the filters. Equation (8) also ensures equation (5) holds.

As stated in section 2, $y(n)$ and $x(n)$ are the highest resolution versions of the original signal $x(t)$. Let the reconstructed continuous time version of the signal be $\hat{x}(t)$. If $y(n)$ is reconstructed from only the $\mathcal{W}_{-1}$ subspace coefficients, then

$$\hat{x}(t) = \sum_n cD_{-1}(n) \frac{1}{\sqrt{2}} \psi\left(\frac{t}{2} - k\right) \tag{11}$$

The error energy between $x(t)$ (given by equation (1)) and $\hat{x}(t)$ is given by

$$E = \int e^2(t)\, dt = \int (x(t) - \hat{x}(t))^2\, dt \tag{12}$$

In order to minimize the projection of the signal onto $\mathcal{W}_{-1}$, the error energy in (12) is to be maximized. This is done by setting the derivative of E with respect to the filter weights to zero.

$$\frac{dE}{dh_1(r)} = 0, \qquad r = 1, \ldots, N \tag{13}$$

Substituting equations (1) and (11), in (12) and solving (13) with the orthogonality conditions, gives the following result

$$\sum_m h_1(m) \sum_n x(2n-m)\, x(2n-r) = 0 \tag{14}$$

where $r = 1 \ldots N$. This result has the form of N linear equation with N unknowns. However, the solution for $h_1(n)$ is constrained by the non-linear equations (3)-(5) and linear equations (6) and (7). Hence, this turns out to be a problem of minimizing a linear cost function (equation (14)) with linear and non-linear constraints.

3.1. Results

The above minimization problem was solved for a given input signal, $x(n)$ (in figure 2(a)), using a technique Sequential Quadratic Programming (SQP) [8]. For an initial guess to the solution, the Daubechies 8 tap high pass filter was used. After obtaining optimal values for $h_1(n)$, equations (8)-(10) were used to calculate the coefficients of the other filter.

Figure 2 shows the decomposition of an audio signal using the Daubechies 8 tap filter bank, and the optimal filter bank. Clearly for the case of the Daubechies filter bank, a lot more energy is present in the cD_{-1} coefficients than in the optimal filter bank. This implies that the cA_{-1} coefficients for the optimal case are more matched to the signal than for the Daubechies case. This can also be seen from figure 2.

If the signal is reconstructed by setting all the cD_{-1} coefficients to zero for the case the Daubechies filter bank, the Mean Squared Error (MSE) between the reconstructed and original signal is 2.647×10^{-6}. For the optimal filter bank, reconstructing the signal with all the cD_{-1} coefficients equal to zero gives $\mathrm{MSE} = 2.868 \times 10^{-7}$.

4. OPTIMIZATION FOR A 2-D SIGNAL

The wavelet transform applied to a 2-D signal is a simple extension of the 1-D case, where the filtering operations are performed along the rows, and then the columns of the signal. Instead of two subbands as in the 1-D case (cA_{-1} and cD_{-1}), four subbands are generated for the 2-D case: LL_{-1}, LH_{-1}, HL_{-1}, and HH_{-1}. The 2-D scaling function and wavelets can be written as the tensor products of the corresponding 1-D functions [2]:

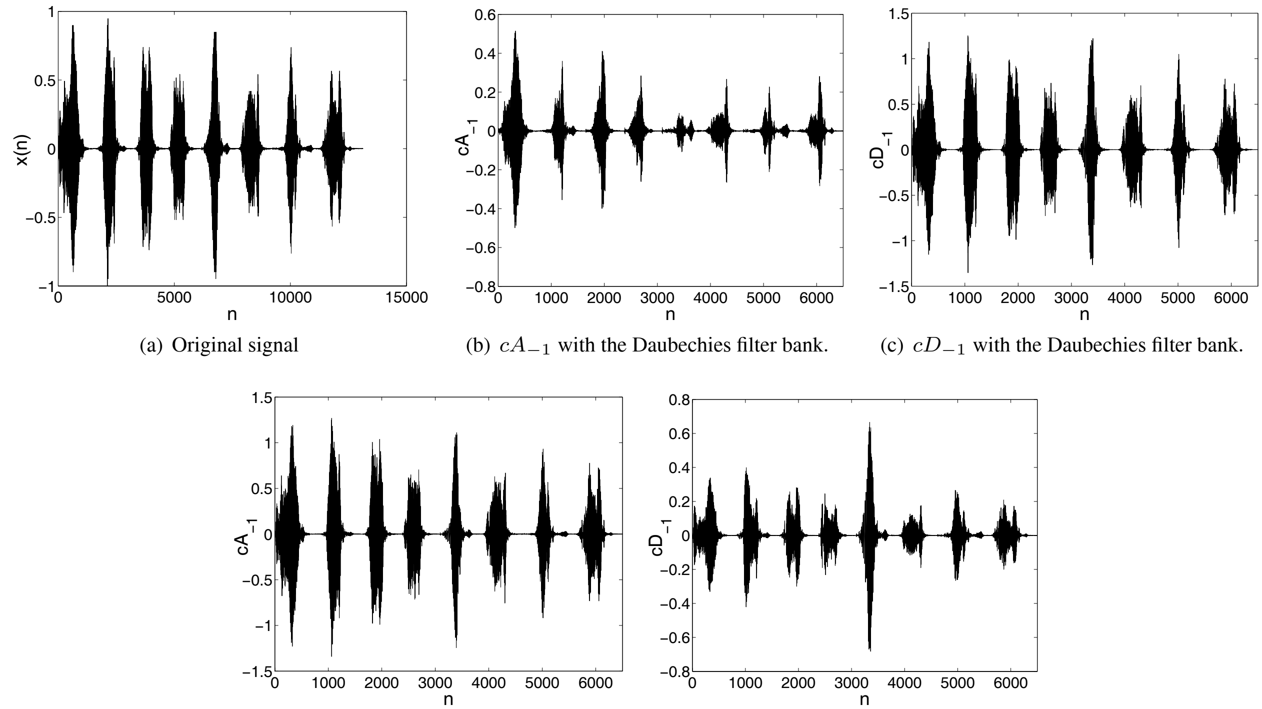
$$\varphi(x,y) = \varphi(x)\varphi(y) \tag{15}$$
$$\psi^H(x,y) = \varphi(x)\psi(y) \tag{16}$$
$$\psi^V(x,y) = \psi(x)\varphi(y) \tag{17}$$
$$\psi^D(x,y) = \psi(x)\psi(y) \tag{18}$$

As in section 3, $x(m,n)$ is the highest resolution version of the signal $x(t_1, t_2)$, and

$$x(t_1, t_2) = \sum_{k_1} \sum_{k_2} x(m,n)\, \varphi(t_1 - k_1, t_2 - k_2) \tag{19}$$

is analogous to equation (1). Similarly, if $\hat{x}(t_1, t_2)$ is reconstructed with only the coefficients from LH_{-1}, HL_{-1}, and HH_{-1}, then

$$\begin{aligned}
\hat{x}(t_1, t_2) = \\
&\sum_{k_1} \sum_{k_2} LH_{-1}(k_1, k_2) \frac{1}{\sqrt{2}} \psi^H\left(\frac{t_1}{2} - k_1, \frac{t_2}{2} - k_2\right) \\
&+ \sum_{k_1} \sum_{k_2} HL_{-1}(k_1, k_2) \frac{1}{\sqrt{2}} \psi^V\left(\frac{t_1}{2} - k_1, \frac{t_2}{2} - k_2\right) \\
&+ \sum_{k_1} \sum_{k_2} HH_{-1}(k_1, k_2) \frac{1}{\sqrt{2}} \psi^D\left(\frac{t_1}{2} - k_1, \frac{t_2}{2} - k_2\right)
\end{aligned} \tag{20}$$



(a) Original signal (b) cA_{-1} with the Daubechies filter bank. (c) cD_{-1} with the Daubechies filter bank.

(d) cA_{-1} with the optimal filter bank. (e) cD_{-1} with the optimal filter bank.

Fig. 2. Decomposition of a signal $x(n)$ using the Daubechies wavelets and the optimized wavelets.

The error energy between the original and reconstructed signal is given by

$$\begin{aligned} E &= \iint e^2(t_1, t_2)\, dt_1 dt_2 \\ &= \iint \left(x(t_1, t_2) - \hat{x}(t_1, t_2)\right)^2 dt_1 dt_2 \end{aligned} \tag{21}$$

Using a similar technique as in section 3 to minimize the projection of the signal onto the wavelet subspaces, the derivative of E with respect to the filter weights is set to zero, as in equation (13).

Solving equation (13) for the 2-D case using equations (15)-(21), and the orthogonality properties, gives:

$$\begin{aligned} \sum_{l_1}\sum_{l_2} h_1(l_1)\, h_1(l_2) \sum_{k_1}\sum_{k_2} x(2k_1 - l_1, 2k_2 - l_2) \cdot \\ x(2k_1 - r, 2k_2 - r) = 0 \end{aligned} \tag{22}$$

where $r = 1 \ldots N$. This is again a minimization problem with N unknowns, but this time, in N non-linear equations. The same linear and non-linear constrains given by equations (3)-(7) apply. This type of system can also be solved using the same SQP technique as mentioned in section 3.1

4.1. Results

The above technique was used to match filters to two test images, the popular Lenna image and an X-ray image. In both cases, $N = 8$, and the initial conditions for the optimization routine were set using the popular Daubechies 8 tap filter bank.

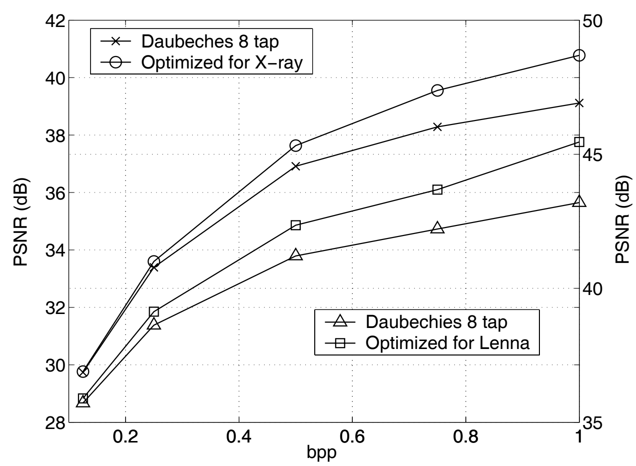
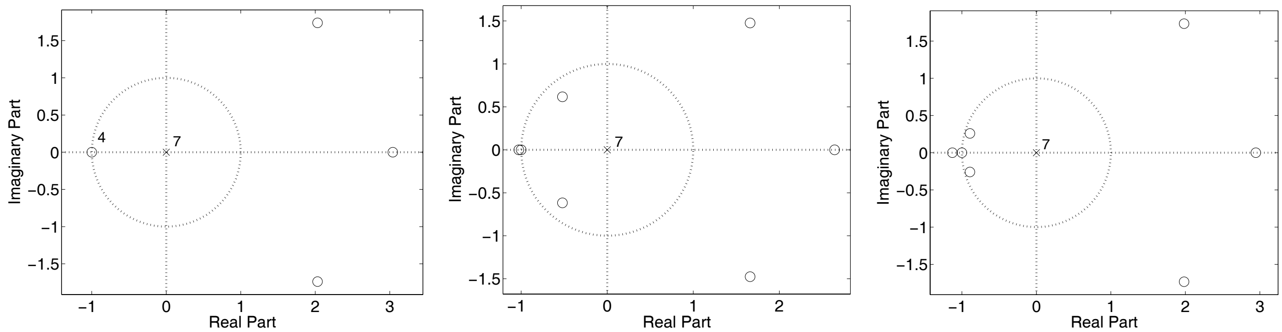


Fig. 3. PSNR vs bpp for the Lenna (bottom) and X-ray (top) test images, coded with the optimized filters as well as the original Daubechies 8 tap filters. The top two graphs follow the right hand side y-axis while the bottom two follow the left y-axis.



(a) Daubechies 8 tap. (b) Optimized for Lenna. (c) Optimized for X-ray.

Fig. 4. Pole-zero plot for $h_0(n)$.

The images were then coded with an implementation of the EZW coding algorithm [3]. The number of decomposition levels was fixed at three. The results for the coding of Lenna and X-ray images are shown in figures 3.

These results show that the filter matched to the image shows a substantial improvement in terms of Peak Signal-to-Noise Ratio (PSNR) at high bpp. At 1 bpp, the PSNR difference between the Daubechies wavelet and the optimized wavelet is approximately 1.5 dB for both images. However, this difference drops off at higher compression ratios. This trend has also been noticed in [9] and [10], where the filters are optimized based on smoothness criteria. This may be explained by the vanishing moments [7, 11] of the wavelets designed, which relates to their approximating power. The vanishing moments are related, but not always equal to the number of zeros at $z = -1$ in $h_0(n)$, see figure 4. The Daubechies functions are designed to have maximum number of vanishing moments (i.e. 4 vanishing moments for filter length of 8). The wavelets designed for the Lenna and X-ray test images have 2 and 1 vanishing moments respectively. As the bit rate reduces, more and more wavelet coefficients are lost due to quantization. This results in the optimized wavelets making coarser and coarser approximations without these coefficients for reconstruction. Because of the lower number of vanishing moments, the rate at which the approximation power is affected with the loss of coefficients is greater than for the case with the Daubechies wavelet.

5. CONCLUSION

The technique described in this paper matches wavelets to 1-D and 2-D signals effectively. Image compression results show a significant PSNR gain of over 1.5 dB at high bit rates. This gain, however, drops off for lower bit rates, which can be explained by the lower approximating power of the optimized wavelets.

It should be made clear that the SQP minimization routine does not give a global minimum solution to the problem. Hence, the initial conditions for the minimization should be considered an important factor for the solution. In tests performed, it was found that for $N = 8$, using the Daubechies filter bank as initial conditions gave the best optimized wavelet.

Future work involves studying further the properties of these optimized wavelet to understand what makes them so effective for that particular image. Another area which is to be investigated is the design of biorthogonal wavelets using the technique described here. Biorthogonal wavelets are less constrained than orthogonal wavelets, and their filters can be symmetric. This potentially allows for an even greater coding gain than for the case of orthogonal wavelets. But in the biorthogonal case, we do not have the orthogonality conditions referred to in section 2.1. This then leads to a more complicated set of equations than (14) and (22).

6. REFERENCES

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