OPTIMAL SUBSAMPLING OF CIRCULARLY BANDLIMITED IMAGES

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ABSTRACT

In this paper we consider the problem of optimal subsampling of circularly bandlimited images. We show that the two facets of this problem can be formulated as constrained sphere packings, and we describe the algorithm that may be used to solve them. We also compare the resulting subsampling patterns to the more conventional rectangular subsampling and illustrate the potential savings in subsampling density.

1. INTRODUCTION

Today even moderately-priced digital cameras produce digital images with resolution in excess of 4 megapixels, and the trend of increasing the resolution continues. While offering excellent quality, such large images are difficult to view in native resolution on most computer monitors. In some cases (e.g. image database browsing) it is necessary to reduce their resolution - this creates the need for efficient subsampling strategies. Subsampling is also important in many other applications, including hierarchical image processing, compression, and hierarchical motion estimation.

Recent work on image subsampling includes hierarchical approaches based on iterative function systems [1] and artificial neural networks [2]. In [1], the authors construct hierarchical subsampling patterns with fractal support, while in [2], an edge-adaptive subsampling approach is proposed. In this work we are interested in the more basic problem of optimal subsampling of a circularly bandlimited image. The assumption of circular frequency support is usually introduced in the absence of prior knowledge about the image, and is a reasonable one for natural images. The problem of optimal subsampling has two facets:

1. For a given subsampling factor, find the subsampling pattern that preserves the largest circular frequency support.
2. For a given circular frequency support, find the subsampling pattern with the highest subsampling factor.

We consider both of these problems and describe an algorithm that may be used to solve them. The paper is organized as follows. In Section 2 we introduce the notation (similar to [3]), review the basics of 2-D subsampling, and show that the two problems above are dual. In Section 3 we describe the algorithm that directly solves the first problem. In Section 4 we illustrate how this algorithm can also be used to solve the second problem, and we compare the resulting subsampling pattern to the more conventional rectangular subsampling. Conclusions are given in Section 5.

2. TWO-DIMENSIONAL SUBSAMPLING

Let $X[\mathbf{n}]$, with $\mathbf{n} = [n_1, n_2]^T$, be a 2-D signal. A subsampled version of X is obtained as $Y[\mathbf{m}] = X[\mathbf{Vn}]$, where $\mathbf{V} = [\mathbf{v}_1|\mathbf{v}_2]$ is a 2×2 integer matrix whose column vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. A subsampling factor is given by $|\det\mathbf{V}|$. In other words, Y retains one out of every $|\det\mathbf{V}|$ samples of X. An illustration of subsampling is shown in the top part of Figure 1, where the samples of X are shown as white dots and the samples of Y are shown as black dots. In the frequency domain, signal Y is represented as a periodic repetition of the frequency content of X (dashed circle), as illustrated in the bottom part of the figure. The periodicity in the frequency domain is specified by the matrix $\mathbf{U} = [\mathbf{u}_1|\mathbf{u}_2]$ whose relationship with the subsampling matrix $\mathbf{V}$ is $\mathbf{U}^T\mathbf{V} = 2\pi\mathbf{I}$. In this text we will consider the frequency normalized by a factor of 2π, so with this normalization the relationship between $\mathbf{U}$ and $\mathbf{V}$ is $\mathbf{U}^T\mathbf{V} = \mathbf{I}$. Hence, $\det\mathbf{U} = 1/\det\mathbf{V}$.

Observe that in order to avoid aliasing, vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ need to be large enough so that the dashed circles in Figure 1 would not overlap. Let us define

$$d_{\min}(\mathbf{U}) = \min\left\{\|\mathbf{u}_1\|, \|\mathbf{u}_2\|, \|\mathbf{u}_1 - \mathbf{u}_2\|, \|\mathbf{u}_1 + \mathbf{u}_2\|\right\}.$$

Hence, $d_{\min}$ is the minimum of the distances between the corner points of a parallelogram whose sides are vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. To avoid aliasing, $d_{\min}(\mathbf{U})$ needs to be larger than the diameter of the circular frequency content of X. Now let $\mathbf{U} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so $\mathbf{V} = (\mathbf{U}^T)^{-1} = \frac{1}{\det\mathbf{U}}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$. Then we have

$$d_{\min}(\mathbf{U}) = \left\{ \begin{array}{c} \sqrt{a^2+c^2}, \sqrt{b^2+d^2}, \\ \sqrt{(a-b)^2+(c-d)^2}, \\ \sqrt{(a+b)^2+(c+d)^2} \end{array} \right\}$$

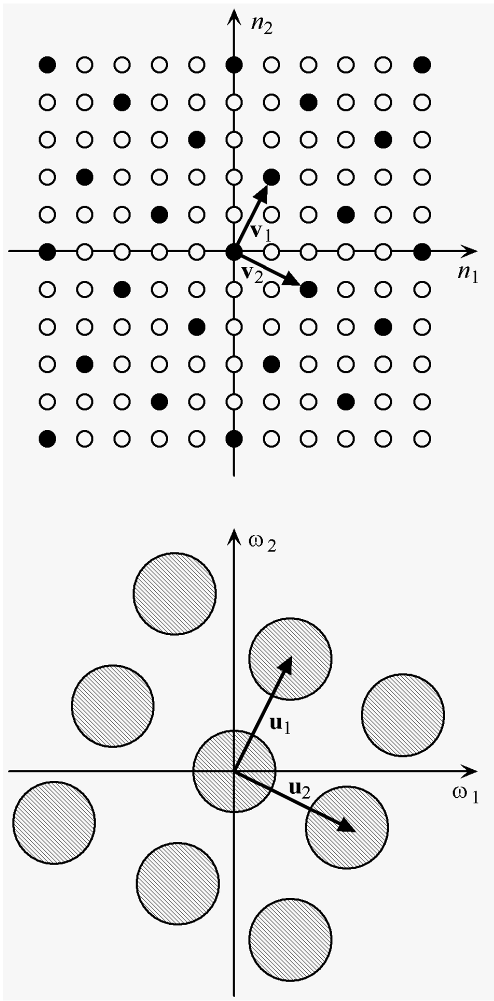


Fig. 1. An illustration of 2-dimensional subsampling in the space domain (top) and the frequency domain (bottom).

and

$$
\begin{aligned}
d_{\min}(\mathbf{V}) &= \frac{1}{|\det \mathbf{U}|} \min \left\{ \begin{array}{c} \sqrt{a^2+c^2}, \sqrt{b^2+d^2}, \\ \sqrt{(a-b)^2+(c-d)^2}, \\ \sqrt{(a+b)^2+(c+d)^2} \end{array} \right\} \\
&= \frac{d_{\min}(\mathbf{U})}{|\det \mathbf{U}|},
\end{aligned}
$$

so for a fixed $|\det \mathbf{V}| = 1/|\det \mathbf{U}|$, maximizing $d_{\min}(\mathbf{V})$ is equivalent to maximizing $d_{\min}(\mathbf{U})$. With this in mind we can rephrase the two problems from Section 1 as follows.

1. For a given $|\det(\mathbf{V})|$, find $\mathbf{V}$ that maximizes $d_{\min}(\mathbf{U})$ or, equivalently, maximizes $d_{\min}(\mathbf{V})$.

2. For a given $d_{\min}(\mathbf{U})$, find $\mathbf{V}$ that maximizes $|\det(\mathbf{V})|$ or, equivalently, minimizes $|\det(\mathbf{U})|$.

Recall that sphere packing [4] seeks to maximize the ratio of the volume of the spheres centered at the corners of the parallelogram described by $\mathbf{U}$ to $|\det(\mathbf{U})|$, i.e. maximize $d^2_{\min}(\mathbf{U})/|\det(\mathbf{U})|$ [5]. Hence, the first problem is equivalent to sphere packing in the space domain, and the second problem is equivalent to sphere packing in the frequency domain. In this sense, the two problems are dual. The solution to the unconstrained sphere packing problem in 2-D is known - it is a hexagonal lattice [4]. However, in our case the samples of X lie on a 2-D integer lattice $\mathbb{Z}^2$, so they can be subsampled only by an integer matrix. Hence, we need to solve a constrained version of the sphere packing problem, which we discuss in the following section.

3. SPHERE PACKING IN $\mathbb{Z}^2$

In our previous work [5] we have formulated an algorithm called ”maximum minimal distance lattice partitioning” that is aimed at solving the sphere packing problem in $\mathbb{Z}^2$. In this section we review this algorithm in the context of 2-D subsampling. For a detailed discussion and comparison with alternative lattice partitioning methods, the reader is referred to [5].

Let P be the desired subsampling factor. For any 2×2 matrix $\mathbf{V}$, let $S_{\mathbf{V}}$ be the set of matrices obtained by rounding the elements of $\mathbf{V}$ to its nearest integers. Any element of $\mathbf{V}$, if not itself an integer, can be rounded up or down, so the set $S_{\mathbf{V}}$ can have up to 16 elements. The algorithm proceeds as follows.

1. Initialization: $a = \sqrt{P/(2\sqrt{3})}$, $\mathbf{V}_a = \begin{bmatrix} 2a & a \\ 0 & a\sqrt{3} \end{bmatrix}$, $\theta = 0$, $d^*_{\min} = 0$.

2. Rotation: $\mathbf{V}_a(\theta) = \mathbf{V}_a \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

3. Check if any $\mathbf{V} \in S_{\mathbf{V}_a(\theta)}$ has both $d_{\min}(\mathbf{V}) > d^*_{\min}$ and $|\det(\mathbf{V})| = P$; if so, set $\mathbf{V}^* = \mathbf{V}$ and $d^*_{\min} = d_{\min}(\mathbf{V}^*)$.

4. Update angle: $\theta = \theta + \Delta\theta$; if $\theta \leq \pi$, go to step **2**, else stop.

After termination, matrix $\mathbf{V}^*$ is the integer matrix with $|\det(\mathbf{V}^*)| = P$ that is the ”most similar” to the hexagonal matrix $\mathbf{V}_a$ that would achieve ideal subsampling. The search for $\mathbf{V}^*$ can be visualized as shown in Figure 2. Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are equal in magnitude and 60° to each other. They are obtained by rotating the column vectors of $\mathbf{V}_a$ by the angle θ. The tips of the vectors show the location of the points of a hexagonal lattice that would achieve ideal subsampling. However, they do not land on the points of the integer lattice. Set $S_{\mathbf{V}}$ contains the matrices whose column vectors are the locations of the integer lattice points closest

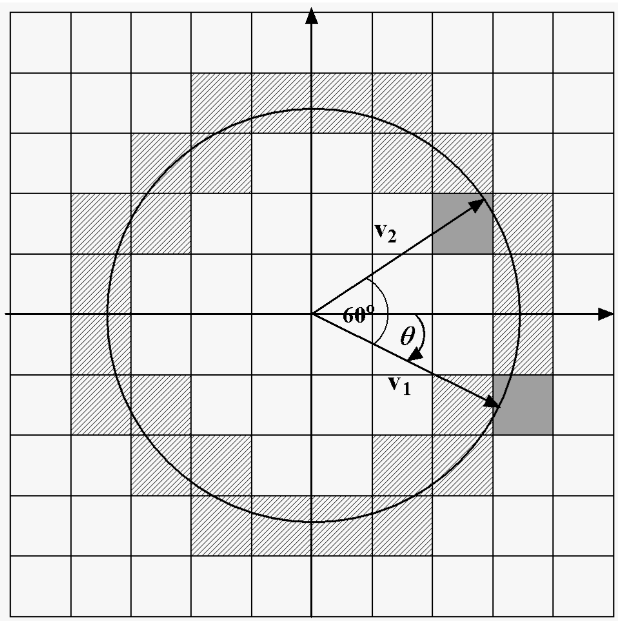


Fig. 2. Illustration of the search for approximately hexagonal patterns in the $\mathbb{Z}^2$ lattice.

to the tips of $\mathbf{v}_1$ and $\mathbf{v}_2$. In Figure 2, they are the corners of the two squares shown in gray. Hence, matrices in $S_{\mathbf{V}}$ are integer matrices that are "similar" to $\mathbf{V}_a(\theta)$ and are potential candidates for the optimal subsampling. The angle increment $\Delta\theta$ is set sufficiently small so that we don't miss any candidate matrices. Details are discussed in [5].

Note that the algorithm is aimed directly at solving the first problem in Section 1, because it seeks to maximize $d_{\min}(\mathbf{V})$ for a given $|\det(\mathbf{V})|$. Examples can be found in [5] and, due to space constraints, we will not repeat them here. In the following section we illustrate how it can also be used to solve the second problem, which may not be obvious from the formulation of the algorithm.

4. AN ILLUSTRATIVE EXAMPLE

Suppose we wish to subsample an image so that we retain the low frequency components in the circular region of diameter 0.27. This can be achieved using a rectangular[1] subsampling matrix

$$\mathbf{V}_r = \begin{bmatrix} 3 & 2 \\ 2 & -3 \end{bmatrix},$$

because its corresponding frequency periodicity matrix is

$$\mathbf{U}_r = (\mathbf{V}_r^T)^{-1} \approx \begin{bmatrix} 0.231 & 0.154 \\ 0.154 & -0.231 \end{bmatrix}$$

[1] Because the column vectors of $\mathbf{V}_r$ are orthogonal and equal in magnitude.

Fig. 3. Lowpass filtered image before subsampling; PSNR = 28.627 dB.

with $|d_{\min}(\mathbf{U}_r)| \approx 0.277 > 0.27$. This corresponds to the subsampling factor of $|\det(\mathbf{V}_r)| = 13$. On the other hand, ideal sampling (by a hexagonal lattice) would introduce a hexagonal periodicity in the frequency domain. One such frequency periodicity matrix is

$$\mathbf{U} = \begin{bmatrix} 0.27 & 0.135 \\ 0 & 0.135\sqrt{3} \end{bmatrix},$$

and others can be obtained from $\mathbf{U}$ by rotation. The ideal sampling factor corresponding to the hexagonal lattice would therefore be $|\det((\mathbf{U}^T)^{-1})| \approx 15.8$, which suggests that retaining one out of 15 samples is sufficient for our purpose. For a subsampling factor of 15, the algorithm from Section 3 (with $\Delta\theta = 10^{-2}$) produces the following matrix:

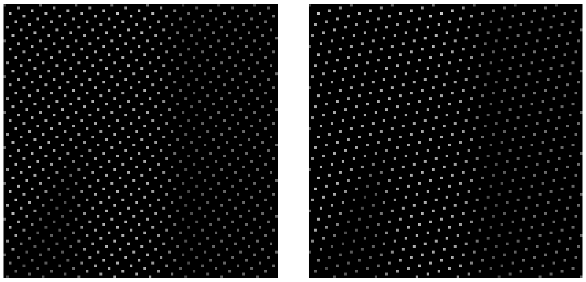
$$\mathbf{V}_h = \begin{bmatrix} 4 & 3 \\ -1 & 3 \end{bmatrix},$$

which corresponds to the frequency periodicity matrix

$$\mathbf{U}_h = (\mathbf{V}_h^T)^{-1} \approx \begin{bmatrix} 0.200 & 0.067 \\ -0.200 & 0.267 \end{bmatrix}$$

with $d_{\min}(\mathbf{U}_h) \approx 0.275 > 0.27$. Hence, $\mathbf{V}_h$ should suffice for our purpose. We compare the two subsampling matrices on the 512×512 grayscale *Lena* image. Prior to subsampling, we filter the image with a 30-th order lowpass Butterworth filter with the cutoff frequency of $0.27/2 = 0.135$ and the kernel size of 31×31 pixels. Our goal here is not to advocate any particular filter, but to illustrate the performance of the two subsampling patterns. The lowpass filtered image is shown in Figure 3.

Since $15/13 \approx 1.15$, the signal subsampled by $\mathbf{V}_h$ has about 15% fewer samples than the signal subsampled by



(a) (b)

Fig. 4. A 100×100 pixel region in the upper left corner of the image showing subsampling patterns: (a) rectangular; (b) approximately hexagonal. From a distance of about 8 - 10 times the height of the figure, the reader should be able to see that the density of samples in (b) is lower than in (a).

$\mathbf{V}_r$. The difference between the two subsampling patterns is illustrated in Figure 4, which shows the upper left corner of the subsampled image. The image is then reconstructed from its subsampled versions by a two-stage lowpass filtering with the same filter as above. The results are shown in Figure 5. Observe that the reconstructed images have almost the same PSNR and are virtually indistinguishable, yet the one in the bottom part of the figure was obtained from 15% fewer samples.

5. CONCLUSIONS

In this paper we have considered optimal subsampling of images with circular frequency support. We have shown that optimal subsampling in this case reduces to two dual problems of constrained sphere packing, and we have described the algorithm that may be used to solve them. Since we are constrained to the $\mathbb{Z}^2$ lattice, the savings of 13.4% in sampling density of an ideal hexagonal lattice with respect to a rectangular lattice [3] cannot always be achieved, because it is not possible to find sublattices of $\mathbb{Z}^2$ that are exactly hexagonal. However, we have demonstrated that savings of this order (or even slightly higher) are possible, virtually without sacrificing the quality of the reconstructed image. The analysis in [5] suggests that it is possible to find infinitely many sublattices of $\mathbb{Z}^2$ that are approximately hexagonal, and in these cases the savings will be around 13.4%. In the future work we will look into extending these concepts to higher dimensions.

Fig. 5. Top: Image reconstructed from rectangular subsampling with a subsampling factor of 13; PSNR = 26.992 dB. Bottom: Image reconstructed from approximately hexagonal subsampling with a subsampling factor of 15; PSNR = 26.974 dB.

6. REFERENCES

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