SVD-BASED IMAGE FILTERING IMPROVEMENT BY MEANS OF IMAGE ROTATION

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ABSTRACT

This article shows that image filtering based on SVD favors the denoising in the line (horizontal) and column (vertical) direction since the matrix SVD is equivalent to a simultaneous line and column vector Principal Component Analysis (PCA). It also proposes a simple algorithm based on PCA-filtering processed on a rotated image such that the principal directions becomes vertical or horizontal. It brings an important improvement in the final image quality since the filtering in every image direction is improved.

1. INTRODUCTION

The Singular Value Decomposition (SVD), or eigenvector analysis, is a classical tool used in two-dimensional grey scale image processing. An image may be decomposed into a sum of rank one matrices (images) that can be termed as eigenimages [1]. The eigenvectors (eigenimages) form an orthogonal basis for the representation of individual images in the image set. Although a large number of eigenvectors may be required for very accurate reconstruction of the image, a smaller number of eigenvectors is generally sufficient to represent the image. This is the basis of the image compression or coding technique known as Karhunen-Loeve transform. Many studies are actually based on an SVD analysis [1, 7, 5].

The SVD is also a classical tool used in a filtering framework to reduce noise in a noisy image. Under the assumption of additive noise, the initial image is usually estimated thanks to a lower rank approximation of the noisy image. This model implies that the largest eigenvalues are associated to the signal components and the lowest eigenvalues to the noise components.

The goal of this paper is to give a deeper analyse of the SVD-based filtering in order to improve the filtering process. Section 2 stresses the equivalence between a matrix Singular Value Decomposition (SVD) and simultaneous Principal Component Analysis (PCA) of line vectors and column vectors. In section 3, a pedagogical example shows that since the SVD analysis an image in the natural lines-columns image arrangement the useful image information is spread over several components and associated eigenvalues (energy). More over, when a signal has a line-parallel or column-parallel spatial distribution, then the useful image information is focused on a low number of components (typically one). In section 4, we propose to use this property to improve the filtering efficiency of a noisy image. It is shown that rotating the image so that the main image orientations becomes parallel to lines (or equivalently to the columns) and applying the classical SVD based filtering improve the filtering in the considered direction. The main reason of this improvement is that the rotation focusses the signal information in a less important number of components and the fewer components describe the signal, the better is the estimated image quality factor which defines the proportion between noise and signal. Finally, section 5 gives a simple algorithm based on successive image rotation and filtering which gives encouraging results in regard to the denoising of a real 2D-image.

2. SVD REVIEW AND INTERPRETATION

Let's first give a brief recall of the Principal Component Analysis process involved in the first order case. Let' consider a set of N-dimensional observation vectors $\{\mathbf{y}_j = (y_{1j}, \ldots, y_{Nj})^T,\ j = 1, \ldots M\}$. The Principal Component Analysis of the set is processed by estimating the covariance matrix of vectors: $C_{YY} = \frac{1}{M}\sum_{j=1}^{M} \mathbf{y}_j \mathbf{y}_j^T = \frac{1}{M}\left[\mathbf{y}_1, \cdots, \mathbf{y}_M\right]\left[\mathbf{y}_1^T, \cdots, \mathbf{y}_M^T\right]^T$. The vector set principal components consist of C_{YY} covariance matrix eigenvectors.

In the second order case, we propose to give a physical interpretation of the matrix SVD process. The classical eigenimage decomposition of a 2D-image (refer to [1, 2]) represents a simultaneous principal component analysis of the lines and of the columns respectively. Let's consider a $(I_1 \times I_2)$ 2D-image represented by matrix $A = \{a_{ij}, i = 1, \ldots, I_1, j = 1, \ldots, I_2\}$. The singular value decomposition of A processed in the classical eigenanalysis of images lead to: $A = USV^T = \sum_{i=1}^{R} \lambda_i A_i$. U (resp. V) is the matrix of matrix A left (resp. right) singular vectors which are also matrix AA^T (resp. $A^T A$) eigenvectors [4]. R is the rank of matrix A, λ_i is its i^{th}-singular value, and $A_i = \mathbf{u}_i \mathbf{v}_i^T$ is the corresponding eigenimage.

Let $\{\mathbf{y}_j = (a_{1j} \dots a_{I_1 j})^T, j = 1, \dots I_2\}$ the I_1- dimensional colum vectors and $\{\mathbf{x}_i = (a_{i1} \dots a_{iI_2})^T, i = 1, \dots I_1\}$ the I_2- dimensional line vectors of matrix A. The analysis of the columns (resp. lines) of image A brings matrix A left (resp. right) singular vectors U (resp. V). Indeed, considering the previous notations we have:

$$AA^T = [\mathbf{y}_1, \cdots, \mathbf{y}_{I_2}] \left[\mathbf{y}_1^T, \cdots, \mathbf{y}_{I_2}^T\right]^T = \sum_{j=1}^{I_2} \mathbf{y}_j \mathbf{y}_j^T,$$

$$A^T A = [\mathbf{x}_1, \cdots, \mathbf{x}_{I_1}] \left[\mathbf{x}_1^T, \cdots, \mathbf{x}_{I_1}^T\right]^T = \sum_{i=1}^{I_1} \mathbf{x}_i \mathbf{x}_i^T, \tag{1}$$

thus, AA^T (resp. $A^T A$) is proportional to the column (resp. line) vector set covariance matrix, up to the multiplicative factor $\frac{1}{I_2}$ (resp $\frac{1}{I_1}$). As a consequence, the left (resp. right) singular vectors of matrix A, which are matrix AA^T (resp. $A^T A$) eigenvectors, are the principal components of the column (resp. line) vector set. In the following, PCA-based filtering and lower rank approximation of a matrix are used equivalently.

3. PROPERTIES OF IMAGE PCA

From the previous section it appears that the SVD determines the principal components of the signal associated to the natural columns and lines image organization. By construction, the principal components are orthogonal which means that their corresponding signal are uncorrelated. As a consequence, any diagonal signal (such as a diagonal line in an image) cannot be detected as a single component, i.e. one singular vector. The information related to this diagonal signal is spread over many components. This property can be visualized in the following pedagogical experiment, and also explains the eigenimage analysis and image compression results of studies [1, 2].

Figure (1-a) shows a diagonal line on a white font. The eigenvalues extracted from image (1-a) SVD are represented in figure (1-c). Each eigenvalue can be physically interpreted as the energy of the corresponding eigenimage. Although the energy associated to the first eigenimage ($3{\cdot}10^4$) is singularly high, the energy of the following eigenimages, that corresponds smaller and smaller image details, remains also high between 10^2 and 10^3, which means the image energy is spread over a large number of eigenimages. This property is verified in image (2-a) in which the lower rank-15 image approximation (i.e. the sum of the 15 eigenimages weighted by the corresponding eigenvalues) does not reconstruct the original image. As can be seen in image (2-b), representing the lower rank-120 image approximation, 120 eigenimages are necessary to obtain a satisfactory reconstructed image.

On the contrary, the information corresponding to vertical image (1-b) is concentrated in the only two first eigenimages. Indeed, as can be seen in figure (1-d), the energy associated to the two first eigenimages are superior to 10^3

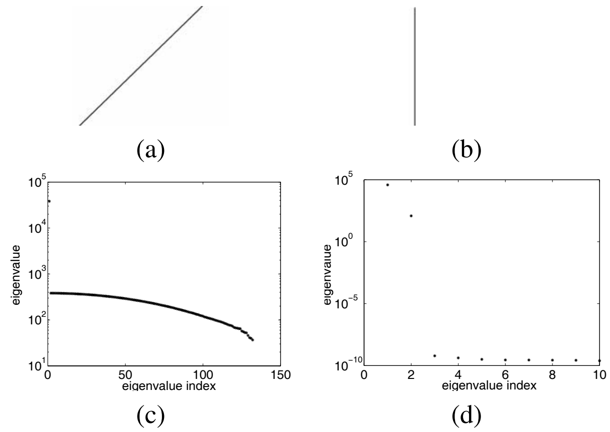


Fig. 1. *Diagonal (a) and vertical (b) lines and corresponding eigenimage energy (c) and (d). Image size:* (132×175)

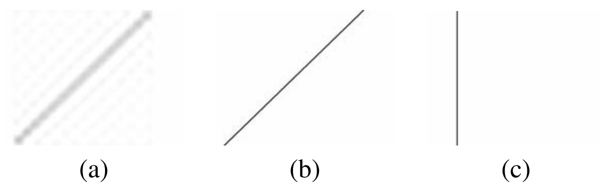


Fig. 2. *Lower rank-15 (a) and rank-120 (b) diagonal line approximation, and lower rank-1 vertical image approximation (c).*

where as the energy associated to the following eigenimages are inferior to 10^{-8}. Thus, the vertical line lower rank-1 approximation, figure (2-c), i.e. the only first principal component, is enough to get a satisfactory reconstructed image.

4. PCA-BASED NOISE FILTERING

Let's make the classical assumption of an additive noise on every image pixel. A noisy $(I_1 \times I_2)$-image B can be considered as the sum of an initial $(I_1 \times I_2)$-image A and a $(I_1 \times I_2)$-image of white Gaussian noise N: $B = A + N$.

In the PCA-based filtering, initial image A is estimated by computing the lower rank-R of noisy image B: $\widehat{A} = \sum_{i=1}^{R} \Lambda_i B_i$.

For $i = 1$ to R, Λ_i is image B i^{th} eigenvalue and B_i is the associated eigenimage. This estimation represents the sum of the R first eigenimages of noisy image B, i.e. the simultaneous line and column projection of B on the R first line and columns principal components. R represents the minimum number of components necessary to get a satisfactory image estimation.

By construction, $\sum_{i=1}^{R} \Lambda_i B_i$ minimize the quadratic distance $\|B - C\|^2 = \mathrm{tr}\left[(B - C)(B - C)^T\right]$ with respect to C, subject to C being a rank-R matrix [3]. Consequently, eigenimage B_i is not a pure signal component, but remains a sum between a signal and a noise component, since B is a noisy image. Indeed, with the assumption that noise N and image A are uncorrelated, the SVD of matrix B leads to: $B = U\Sigma V^T$, with $\Sigma = \text{diag}(\Lambda_1, \ldots, \Lambda_P)$, and $\Lambda_i = \lambda_i + \sigma_i^2$, $i = 1, \ldots I_{min}$. λ_i and σ_i^2 are the i^{th}-component signal and noise energy, and $I_{min} = \min(I_1, I_2)$. Thus, matrix B eigenvalues represent the sum between the signal and the noise energy.

According to the previous remark, we define a quality factor Q which enables us to quantify the proportion of noise and signal in the estimated image $\widehat{A}$, given by $Q(\widehat{A}) = \frac{\sum_{i=1}^{R} \lambda_i}{\sum_{i=1}^{R} \sigma_i^2}$. The higher Q is the less noise is present in the estimation, and thus, the better is the filtering quality. Considering a white noise, i.e. $\sigma_1^2 = \ldots \sigma_N^2 = \sigma^2$, the quality factor of estimation $\widehat{A}$ becomes:

$$Q(\widehat{A}) = \frac{1}{R\sigma^2} \sum_{i=1}^{R} \lambda_i. \tag{2}$$

As a consequence, one means to increase the filtering quality, is to reduce the number R of components on which the signal energy is distributed, since according to (eq. 2), supposing a constant signal energy $\sum_{i=1}^{R} \lambda_i$, a low R implies a high estimation quality factor $Q(\widehat{A})$.

This property is verified in the previous examples of section 3, in which the energy of the diagonal line (image 1-a) is spread over 120 components (image 2-b) where as the energy of the vertical line (image 1-b) is only concentrated on the first component (image 2-c). Some additive white Gaussian noise is added to the image of the diagonal line (3-a) and the vertical (3-b) line, with a SNR=-1dB. According to image 1-d, the filtering quality of the lower rank-1 approximation of the noisy vertical line is logically better than the lower rank-50 approximation of the noisy diagonal line since the number of components involved in the vertical line description is lower than the one involved in the diagonal line description.

5. SVD AND ROTATION BASED IMAGE FILTERING

As shown in section 4, the fewer components describe the signal, the better is the estimation quality factor ($Q(\widehat{A})$). In section 3, it is also shown that the more the principal directions of an image is aligned with the natural lines or columns directions (horizontal or vertical), the lower is the number of components necessary to describe the signal. The signal energy is focussed on a reduced number of components.

One simple means to make the principal directions of an image (A, fig. 3-a) be vertical or horizontal is to proceed to an appropriate image rotation. The resulting rotated image is comprised in a bigger rectangular image (B, fig. 4-a), in which the pixels that does not match to the rotated image are filled with zero values. The lower rank approximation of image B enables to extract image A principal direction with a very low number of components. Nevertheless, the number of components is still larger than one because of rotated image border effects within B. The effect of the rotation on the initial image is to focuss the useful information on a lower number of components. This focussing increases the PCA-based filtering estimation quality factor Q, and thus increase the image filtering quality.

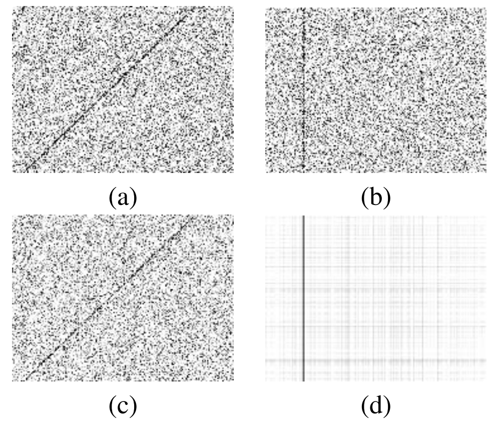


Fig. 3. *Image of noisy diagonal (a) and vertical (b) lines (SNR=-1dB). Lower rank-50 approximation of noisy diagonal line (c) and lower rank-1 approximation of noisy vertical line (d).*

NQE for:	PCA filtering	Rotation-based filtering
Diagonal line	0.19	0.13
Lena	0.27	0.1

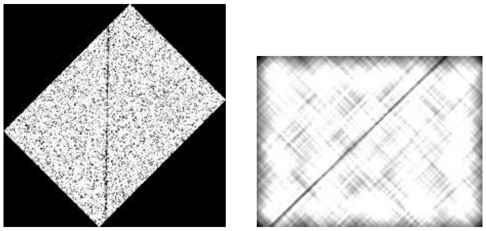
Table 1. *NQE involved in PCA-based and SVD-and-rotation-based filtering*

In order to a posteriori verify the estimated image quality we propose to use the Normalized Quadratic Error criterion (NQE) defined thanks to the matrix Frobenius Norm by:

$$NQE(\widehat{A}) = \frac{\|\widehat{A} - A\|^2}{\|A\|^2}. \tag{3}$$

The filtering quality of image 4-b, obtained by lower rank-7 approximation of matrix B (NQE=0.13) is better than the classical lower rank-50 of the non-rotated image (given image 3-c, with NQE=0.19), which is confirmed by the respective NQE given in table 1.

However, in an unknown noisy image, the principal directions, if ever they exist, are also a-priori unknown. We have shown that rotating an image and applying the PCA-based filtering improve the filtering quality in the particular direction of the rotation. Thus, one simple algorithm to improve the filtering quality in every direction of the image consists in processing the PCA-based filtering for a set of rotation angles comprised between 0° and 360°, and in av-



(a) (b)

Fig. 4. *(a): 45 rotation of noisy image 3-a. (b): rank-*7 *approximation of image (a).*

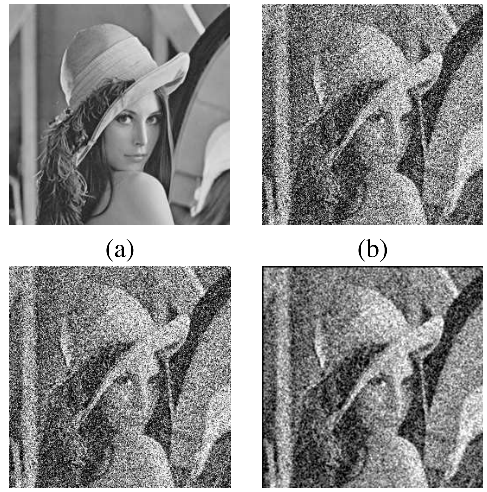
eraging the different estimations in a final estimation image. We expect the averaging of the images obtained after filtering the initial image along several directions to improve the final filtering compared to the one obtained on the initial non rotated image.

This algorithm is tested on noisy image 5-b resulting from the addition of white Gaussian noise on "Lena" standard image 5-a (SNR=-1.26dB). The classical PCA-based filtering processed by lower rank-100 of image 5-b is given in image 5-c (NQE=0.27). The new PCA-rotation based filtering applied on image 5-b gives better results than the classical PCA-based method (image 5-d, with NQE=0.1). The good quantitative results of this statement can be verified thanks to the NQE computed for the two methods, and summarized in table 1. In this simulation a set of 180 rotation angles is used, and obtained by proceeding 179 consecutive $2°$-rotations from $0°$ to $358°$.

Note that this rotation-based algorithm can be generalized to multidimensional data such that color and multispectral images which are modelled by higher order tensors, and for which new multidimensional and multi-way filtering methods have recently been developed [6].

6. CONCLUSION

In this article, it is first recalled the equivalence between a matrix Singular Value Decomposition (SVD) and simultaneous Principal Component Analysis (PCA) of line vectors and column vectors. Then, it is shown that due to the natural arrangement of matrices in lines and columns, the SVD favors a vertical and horizontal direction of analysis in an image. As a consequence, a classical PCA-based filtering of a noisy image emphasizes the denoising on the vertical and horizontal directions. It is also shown that the more the principal directions of an image is aligned with the natural lines and columns directions (horizontal or vertical), the lower is the number of components necessary to describe the signal since the signal energy is focused on a reduced number of components. Moreover, in a filtering framework, the fewer components describe the signal, the better is the estimation quality factor ($Q(\widehat{A})$). Thus, a simple algorithm based on PCA-filtering processed on rotated image such that some particular image directions becomes vertical or horizontal is elaborated and brings an important improvement in the final image quality since the filtering in every image direction is improved.



(a) (b)

(c) (d)

Fig. 5. *(a): Lena standard* (256×256)*-image.(b): Noisy image (SNR=-1.26dB). (c): Lower rank-100 approximation of noisy image (b). (d): Mean of lower rank-100 approximation processed on consecutive* $2°$*-rotated images.*

7. REFERENCES

[1] H.C. Andrews and C. L. Patterson. Sing. val. decomp. and digital im. proc. *IEEE Trans. on ASSP*, 24(1):26–53, february 1976.

[2] H.C. Andrews and C. L. Patterson. Sing. val. decomp. (svd) im. coding. *IEEE Trans. on Com.*, pages 425–432, april 1976.

[3] Y. Eckart and G. Young. The approx. of a matrix by another of lower rank. *Psychometrika*, 1:211–218, 1936.

[4] G.H. Golub and C.F. Van Loan. *Matrix comp.* 3rd ed. Baltimore, the john hopkins univ. press edition, 1996.

[5] Murase H. and Lindenbaum M. Combined tech. of sing. val. dec. and vect. quantization for im. coding. *IEEE Trans. Im. Proc.*, 4(5):620–629, May 1995.

[6] D. Muti and S. Bourennane. Multidim. signal proc. using lower rank tensor approx. In *IEEE ICASSP03*, Hong Kong, China, April 6-10, 2003.

[7] Kakarala R. and Ogumbona P.O. Sig. anal. using a multiresolution form of the sing. val. dec. *IEEE Trans. Im. Proc.*, 10(5):724–735, may 2001.