DISCRETE SPACE MODELS FOR SELF-SIMILAR RANDOM IMAGES

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ABSTRACT

Images exhibiting statistical self-similarity are of interest in various areas of image processing such as textures and scene synthesis. In continuous-space, statistical self-similarity is defined through statistics invariant to spatial scaling. However, because of lack of discrete-space scaling operation, statistical self-similarity in discrete-space has been characterized by approaches such as increments of fractional Brownian motion rather than scaling. We address these two issues regarding self-similar random fields through the paper. We show that the current self-similarity definition for continuous-space is somewhat restrictive, and we offer a new self-similarity definition in continuous-space more general than the current one. Furthermore, we provide a new formalism for statistical self-similarity in discrete-space by defining a scaling operation for discrete-space images. Consequently, a wider class of self-similar random images can be synthesized for different applications in discrete-space. The paper presents theoretical development and synthesis examples.

1. INTRODUCTION

Statistical self-similarity in images has received attention for its ability to describe certain types of natural patterns, which are not described accurately by other mathematical models. Applications of such self-similar images include segmentation/classification of objects in remote sensing images [1], diagnosis through medical imaging [2], synthesis of natural scenes [3] and classification and segmentation of texture images [4].

The currently used definition of statistical self-similarity [5] involves *isotropic scaling*, or scaling of both axes by the same factor. However, as we show in this paper, this definition for self-similarity is not general enough, and other random fields that would be justified as self-similar in some sense are not covered by it. Yet another problem we address is that a counterpart to this continuous-space definition does not exist in discrete-space, and hence, for digital images. Therefore, other approaches such as stationary increments of the fractional Brownian motion [6, 7, 8] have been proposed for discrete-space self-similar random fields. An approach to defining self-similarity in discrete-space using a discrete-domain scaling operator was proposed by Zhao and Rao [9, 10], However, as this paper shows, that approach is also restrictive.

The paper rectifies these problems (1) by providing a definition for statistical self-similarity in continuous-space that is more general than the current definition and (2) by developing a formalism for self-similarity based on scaling that works in discrete-space rather than continuous-space.

The paper is organized as follows. Section 2 proposes a new definition of the generalized self-similarity for random images in continuous space. Section 3 provides a formulation which conducts the scaling operation in discrete-space and a new definition for wide-sense discrete self-similar random fields. The algorithm to generate the discrete-space self-similar images and synthesize examples are provided in Section 4. Concluding remarks are drawn in Section 5.

2. GENERALIZED SELF-SIMILARITY IN CONTINUOUS SPACE

Samorodnitsky and Taqqu [5] define a statistically self-similar random field with Hurst parameter H as a random field $x(\mathbf{t})$ satisfying

$$x(a\mathbf{t}) \stackrel{d}{=} a^H x(\mathbf{t}), \quad a > 0, \tag{1}$$

where $\mathbf{t} = [t_1, t_2]^T$, and $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions. However, the definition, which requires the same scaling in each coordinate, is somewhat restrictive. For example, suppose a random field $h(t_1, t_2)$ is composed of two independent 1-D random processes $f(t)$ and $g(t)$ as

$$h(t_1, t_2) = f(t_1)g(t_2) \tag{2}$$

where only $g(t)$ is self-similar with Hurst parameter H_g. Then $h(t_1, t_2)$ satisfies

$$h(t_1, at_2) \stackrel{d}{=} a^{H_g} h(t_1, t_2). \tag{3}$$

Clearly, $h(t_1, t_2)$ is directionally self-similar. However, such self-similarity is not accommodated by the definition in (1). The problem with the definition in (1) is that it is a direct adoption of the 1-dimensional definition of self-similarity, and the additional degree of freedom obtained by moving to 2 dimensions is not used. We now offer an alternative definition based on matrix scaling that proves to be more general.

Definition 1 *A random field $x(\mathbf{t})$ is self-similar for a matrix class C with index H if, for a non-singular matrix $\mathbf{A} \in C$,*

$$x(\mathbf{At}) \stackrel{d}{=} |D|^{H/2} x(\mathbf{t}) \tag{4}$$

where $D = \det \mathbf{A}$.

This work was supported in part by a grant from ITT corporation.

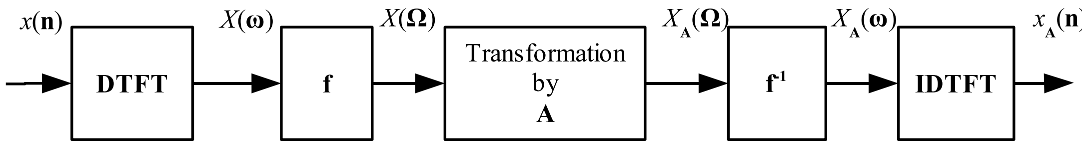

Fig. 1. Block diagram of the 2-D generalized scaling operation

It is now seen that the definition in (1) is a special case of the new definition and holds for the class of diagonal matrices with equal entries, i. e., matrices of the form

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \quad a > 0. \tag{5}$$

On the other hand, the directionally self-similar random field $h(\mathbf{t})$ in (3) is a self-similar random field with respect to matrices of the class given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \quad a > 0. \tag{6}$$

3. STATISTICAL SELF-SIMILARITY IN 2 DIMENSIONAL DISCRETE-SPACE

We now formulate a scaling operator in discrete-space that leads to developing a framework for treating self-similarity on lines analogous to that for continuous space as in (4). Armed with such a formalism for discrete-space, we can address the issue of self-similarity in digital images.

Let $f(\omega)$ be a 1-D warping function that transforms a frequency $\omega \in [-\pi, \pi]$ to $\Omega \in [-\infty, \infty]$, where ω may be regarded as the frequency variable in the discrete time Fourier transform of a discrete time signal while Ω is the same for the continuous time Fourier transform of a continuous time signal [11]. Then a 2-D frequency warping transform $\mathbf{f}(\boldsymbol{\omega})$ is a vector valued function

$$\boldsymbol{\Omega} \triangleq \mathbf{f}(\boldsymbol{\omega}) = [f(\omega_1)\, f(\omega_2)]^T, \tag{7}$$

where $\boldsymbol{\Omega} = [\Omega_1, \Omega_2]^T$ and $\boldsymbol{\omega} = [\omega_1, \omega_2]^T$. Its inverse function $\mathbf{f}^{-1}(\boldsymbol{\Omega})$ maps $\boldsymbol{\Omega}$ to $\boldsymbol{\omega}$. We define the scaling operation $\mathcal{T}_{\mathbf{A}}[\,]$ in 2-D discrete-space as

$$\mathcal{T}_{\mathbf{A}}[x(\mathbf{n})] \triangleq \mathcal{G}^{-1}\left[|D| X(\boldsymbol{\Lambda}_{\mathbf{A}}(\boldsymbol{\omega}))\right], \tag{8}$$

where $\mathbf{A}$ is a 2×2 matrix,

$$\boldsymbol{\Lambda}_{\mathbf{A}}(\boldsymbol{\omega}) \triangleq \begin{bmatrix} \Lambda_1(\omega_1, \omega_2) \\ \Lambda_2(\omega_1, \omega_2) \end{bmatrix} = \mathbf{f}^{-1}[\mathbf{A}^T \mathbf{f}(\boldsymbol{\omega})], \tag{9}$$

and $\mathcal{G}^{-1}$ is the inverse discrete-space Fourier transform

$$\mathcal{G}^{-1}[X(\boldsymbol{\omega})] = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\boldsymbol{\omega}) e^{j\boldsymbol{\omega}\cdot\mathbf{n}}\, d\boldsymbol{\omega}. \tag{10}$$

The procedure of the transformation is summarized in Fig. 1.

We now define discrete-space statistical self-similarity using the scaling operator $\mathcal{T}_{\mathbf{A}}$.

Definition 2 *A discrete-space random field $x(\mathbf{n})$ is self-similar with degree H in wide-sense, for a class of matrices C, if it satisfies*

$$\begin{aligned} E[\mathcal{T}_{\mathbf{A}}[x(\mathbf{n})]] &= |D|^{-H/2} E[x(\mathbf{n})] \\ \mathcal{T}_{\mathbf{A}\mathbf{A}}[R_{xx}(\mathbf{n}, \mathbf{n}')] &= |D|^{-H} R_{xx}(\mathbf{n}, \mathbf{n}') \end{aligned} \tag{11}$$

for a non-singular matrix $\mathbf{A} \in C$, where $R_{xx}(\mathbf{n}, \mathbf{n}')$ is the autocorrelation of the image $x(\mathbf{n})$, and $D = \det \mathbf{A}$.

The definition reduces to the discrete-space self-similar definition in [9] if the transform matrix is a diagonal matrix of the form of $a\mathbf{I}$ for scalar $a > 0$ and identity matrix $\mathbf{I}$.

For a zero-mean stationary random field $x(\mathbf{n})$ with a power spectrum $P_x(\boldsymbol{\omega})$, the self-similar definition (11) can be simplified as

$$\frac{P_x[\boldsymbol{\Lambda}_{\mathbf{A}}(\boldsymbol{\omega})]}{\left|\det\left[\frac{d\boldsymbol{\Lambda}_{\mathbf{A}}(\boldsymbol{\omega})}{d\boldsymbol{\omega}}\right]\right|} = |D|^{-H-2} P_x(\boldsymbol{\omega}). \tag{12}$$

where

$$\frac{d\boldsymbol{\Lambda}_{\mathbf{A}}(\boldsymbol{\omega})}{d\boldsymbol{\omega}} \triangleq \begin{bmatrix} \frac{\partial \Lambda_1(\omega_1,\omega_2)}{\partial \omega_1} & \frac{\partial \Lambda_2(\omega_1,\omega_2)}{\partial \omega_1} \\ \frac{\partial \Lambda_1(\omega_1,\omega_2)}{\partial \omega_2} & \frac{\partial \Lambda_2(\omega_1,\omega_2)}{\partial \omega_2} \end{bmatrix} \tag{13}$$

One example of such a self-similar random field is a zero-mean wide sense stationary random field with a power spectrum

$$P(\boldsymbol{\omega}) = \frac{\|\mathbf{f}(\boldsymbol{\omega})\|^r}{\left|\det\left[\frac{d\,\mathbf{f}(\boldsymbol{\omega})}{d\,\boldsymbol{\omega}}\right]\right|}. \tag{14}$$

It can be shown such a random field is wide-sense self-similar with $H = -\frac{r}{2} - 1$ with respect to

$$\mathbf{A} = \alpha \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{15}$$

With the bilinear warping transform (BLT) $f(\omega) = 2\tan(\omega/2)$, the power spectrum (14) becomes

$$P(\boldsymbol{\omega}) = \frac{2^r\left[\tan^2(\omega_1/2) + \tan^2(\omega_2/2)\right]^{r/2}}{|\sec^2(\omega_1/2)\sec^2(\omega_2/2)|}. \tag{16}$$

This power spectrum with $r = -1.4$ ($H = -0.3$) is depicted in Fig. 2.

4. SYNTHESIS OF DISCRETE-SPACE SELF-SIMILAR RANDOM FIELDS

Unlike the 1-D case, which utilized the factorization of the 1-D power spectrum to construct a linear filter [11], 2-D factorization is usually not available. An indirect approach to achieve factorization

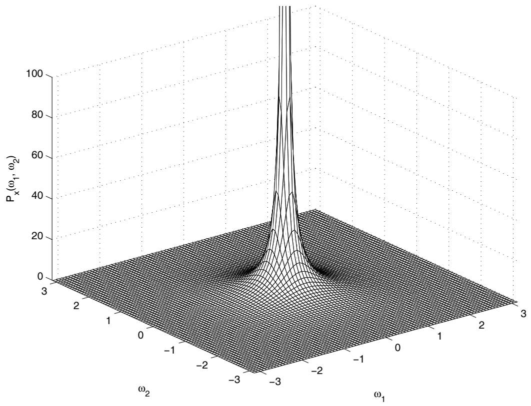

Fig. 2. Power spectrum (16) with $r = -1.4$

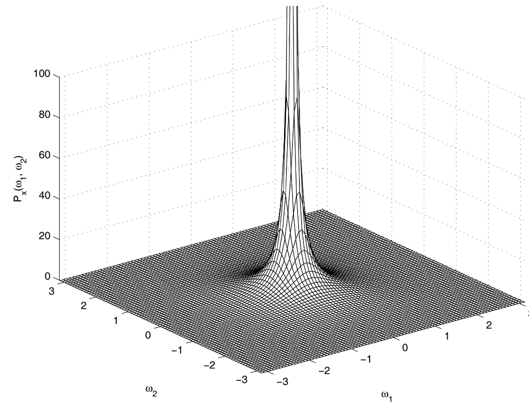

Fig. 3. Modified power spectrum (23) with $r = -1.4$

of a 2-D power spectrum is to utilize the *(complex) cepstrum* of the autocorrelation of a random field. The cepstrum $\hat{x}(\mathbf{n})$ of a discrete image $x(\mathbf{n})$ is defined as a 2-D homomorphic transform [12, 13]

$$\hat{x}(\mathbf{n}) \triangleq Z^{-1}[\ln Z[x(\mathbf{n})]] \tag{17}$$

where Z and Z^{-1} represent forward and inverse Z-transform. The original image $x(\mathbf{n})$ is obtained by

$$x(\mathbf{n}) = Z^{-1}\left[\exp[Z[\hat{x}(\mathbf{n})]]\right]. \tag{18}$$

Let $P_x(\mathbf{z})|_{\mathbf{z}=e^{j\boldsymbol{\omega}}}$ be a power spectrum of $x(\mathbf{n})$ and composed of two factors as

$$P_x(\mathbf{z}) = B_+(\mathbf{z})B_-(\mathbf{z}) \tag{19}$$

Then in cepstrum space (19) becomes

$$\hat{R}_x(\mathbf{n}) = \hat{b}_+(\mathbf{n}) + \hat{b}_-(\mathbf{n}). \tag{20}$$

where $\hat{R}_x(\mathbf{n})$ is the cepstrum of the autocorrelation $R_x(\mathbf{n})$. The inverse cepstrum $b_+(\mathbf{n})$ from a properly chosen $\hat{b}_+(\mathbf{n})$ can lead to a stable half-plane recursive filter. Then an output $x(\mathbf{n})$ with the power spectrum $P_x(\boldsymbol{\omega})$ of a white noise input $w(\mathbf{n})$ to the recursive filter $b_+(\mathbf{n})$ computed by

$$x(\mathbf{n}) = \sum_{\mathbf{n}} \delta(\mathbf{k})w(\mathbf{n}-\mathbf{k}) - \sum_{\mathcal{R}_+-(0,0)} b_+(\mathbf{k})x(\mathbf{n}-\mathbf{k}) \tag{21}$$

where $\mathcal{R}_+$ is the non-symmetric half plane and is defined as

$$\mathcal{R}_+ \triangleq \{n_1 \geq 0, n_2 \geq 0\} \cup \{n_1 < 0, n_2 > 0\}. \tag{22}$$

We construct a recursive filter from the power spectrum (16) using the cepstrum approach . However, since the power spectrum in (16) contains zero and infinity values, the power spectrum is modified slightly by adding small constants to compute the cepstrum. In the case of the BLT, the modified power spectrum is

$$\tilde{P}(\omega) = \frac{2^r[\tan^2(\frac{\omega_1}{2}) + \tan^2(\frac{\omega_2}{2}) + c_1]^{r/2}}{\sec^2(\frac{\omega_1}{2})\sec^2(\frac{\omega_2}{2})} + c_2 \tag{23}$$

where $c_1,\, c_2 \ll 1$. Fig. 3 depicts the modified power spectrum (23) with $r = -1.4$. Compared with the original power spectrum in Fig. 2, the plot shows the two to be very close.

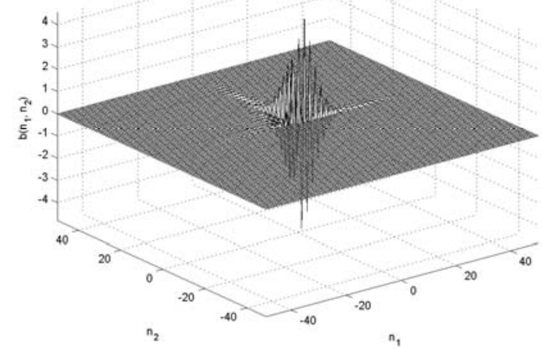

Fig. 4. Filter $b_+(\mathbf{n})$ with $r = -1.4$

Fig. 4 shows the generated filter $b_+(\mathbf{n})$ when $r = -1.4$. Filter taps between $[-50, 49] \times [-50, 49]$ are shown in the plots. To generate a discrete self-similar field, a white noise input is applied to the filter using recursive filtering in (21). Fig. 5 shows an image synthesized from the filter.

Fig. 6 shows an example of a directional self-similar random field. The power spectrum used for this random field is

$$P(\omega_1, \omega_2) = \frac{|f(\omega_1)|^r}{|f'(\omega_1)|}g(\omega_2) \tag{24}$$

where $f(\omega)$ is the BLT, and $g(\omega) = 0.5\,(1 + \cos(\omega))$. It can be shown that the random field is self-similar with $H = -r - 1$ with respect to

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \quad a > 0. \tag{25}$$

For this example, $r = -1.2$ is used.

5. CONCLUSION

This paper presented new definitions for self-similar random fields in continuous and discrete-space. We showed that the currently used self-similarity definition in continuous-space is too restrictive

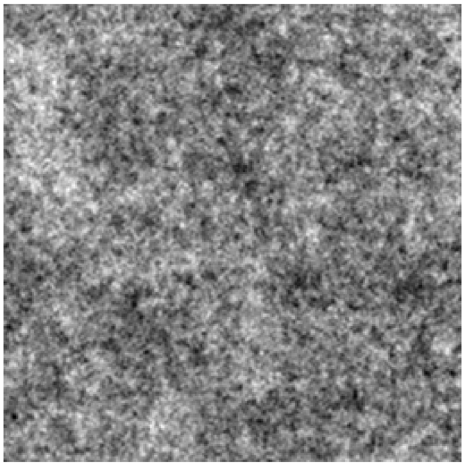

Fig. 5. Synthesized discrete self-similar image with $r = -1.4$

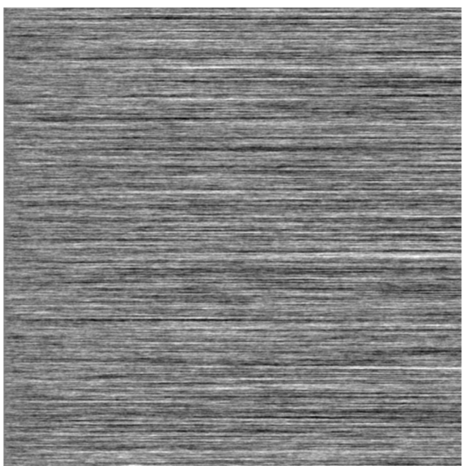

Fig. 6. Directional self-similar image with $r = -1.2$

and does not cover types of random fields which can be considered self-similar in some sense. The new self-similarity definition for continuous-space was proposed based on a scaling operation by a matrix. Through the new definition, it was possible to express a wider class of self-similar random fields in continuous-space. The lack of a scaling operation in discrete-space has been resolved by defining a new scaling operation for discrete-space using warping. This discrete-space scaling was utilized to define discrete-space self-similarity in a way similar to the continuous-case, and it was also possible to define a wider class of self-similar random fields in discrete-space. We also provided an algorithm to synthesize discrete-space random fields from white noise input. Some examples of images synthesized by the method were provided. Potential application of the discrete-space random fields include synthesis of realistic self-similar textures and natural scenes.

6. REFERENCES

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