AN APPROACH BASED ON INFLUENCE FUNCTION TO EVALUATE ROBUSTNESS AND DETECTION PERFORMANCE OF CFAR DETECTORS

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ABSTRACT

A method for evaluating both the robustness and detection performance of constant false alarm rate (CFAR) detectors is presented in this paper, which is based on the powerful methodology of influence function (IF) developed in the literatures on robust statistic. The robustness of different kinds of CFAR detectors can be evaluated and compared by calculating the first derivative of the false alarm probability (FAP) at an underlying distribution, which is named IF-FAP. From the analysis of IF-FAP, several conclusions are given here: the robustness of detector can be asymptotically represented by the IF of clutter power estimator, while the detection performance can be approximately evaluated by the asymptotic variance of clutter power estimator which is an explicit functional of the corresponding IF.

1. INTRODUCTION

Constant False Alarm Rate (CFAR) radars employ adaptive threshold techniques for automatic signal detection. The Cell Averaging (CA) CFAR detectors perform optimally in uniform Rayleigh clutter. The Ordered Statistic (OS) CFAR methods have been proven to work satisfactorily in both multiple-target and nonuniform-clutter situations [1][2]. These detectors achieve CFAR while the distribution parameters vary. However, as well known, if the clutter distribution deviates from presumption, their performance will degrade, and the false alarm probability will not be constant anymore. To solve these problems, some robust CFAR methods have been proposed and discussed in recent literatures [3][4]. "Robustness" is often characterized by the performance of detection in inhomogeneous clutter, or by the variation extent of false alarm probability and CFAR loss. However, robustness is usually referred to as a vague concept due to the lack of quantitative measures. This paper addresses this problem and proposes an approach based on Influence Function (IF) to evaluate robustness and detection performance of CFAR detectors.

Influence function was invented by Hampel in [5]. It is widely used to evaluate the robustness of statistics. It is defined by the derivative of a statistic at an underlying distribution:

$$IF(x;\theta,F)=\lim_{t\downarrow 0}\frac{\theta((1-t)F+t\Delta_x)-\theta(F)}{t} \quad (1)$$

where Δ_x is the probability measure which puts mass 1 at the point $x(x\in\mathbb{R})$, and $\theta(F)$ is the asymptotic value of $\{\theta_n(X_1,\cdots,X_n);n\geq 1\}$, $X_1,\cdots,X_n$ are the observations with distribution F. The importance of IF lies in its heuristic interpretation: one could say it gives a picture of the infinitesimal behavior of the asymptotic value, so it measures the asymptotic bias caused by contamination in the observations [5].

The IF was introduced into radar applications to measure the variation gradient of False Alarm Probability (FAP) caused by the disturbance on clutter distributions, which is named IF-FAP [6]. Some basic results were provided in [6], including the IF-FAP of CA- and OS-CFAR detectors and a computational simulation to verify the theoretical results. Further results are presented in Sect. 2.1.

From the robustness point of view, Hampel introduced some important summary values of the IF [5]. The most important one is the supremum of the absolute value, which was defined by

$$\gamma^*(\theta,F)=\sup_x|IF(x;\theta,F)|, \quad (2)$$

the supremum being taken over all x where $IF(x;\theta,F)$ exists. This value is named the *gross-error sensitivity*, measures the worst influence that a small amount of contamination of fixed size can have on the estimator. Moreover, an estimator minimizing γ^* is called *most B-robust*. The concept of most B-robust is used in Sect. 2.2 to find the most robust one among some certain kind of CFAR detectors.

From the efficiency point of view, Hampel discussed the *asymptotic variance* of estimator which is closely related to IF by

$$V(\theta,F)=\int IF(x;\theta,F)^2dF(x). \quad (3)$$

Under the assumption of asymptotical normality, the asymptotic variance $V(\theta,F)$ of $\{\theta_n;n\geq 1\}$ at F is given in

$$\mathcal{L}_F(\sqrt{n}(\theta_n-\theta(F))\stackrel{\text{weakly}}{\longrightarrow}N(0,V(\theta,F)) \quad (4)$$

when $n \to \infty$, where $\mathcal{L}_F$ means "the distribution of ... under F". It is proposed in Sect. 3 that there is a close relationship between the asymptotic variance of clutter power estimator and the detection performance of CFAR detector.

2. ROBUSTNESS EVALUATED BY IF-FAP

2.1. Robustness Measures Derived from IF-FAP

If we replace the statistic θ by the false alarm probability P_{fa} in the radar application, the corresponding IF-FAP is given by

$$IF(x; P_{fa}, F) = \lim_{t \downarrow 0} \frac{P_{fa}((1-t)F + t\Delta_x) - P_{fa}(F)}{t} \tag{5}$$

which describes the effect on the FAP variety of an distribution contamination. It forms an influence curve when x takes the value along the real axis. The ability of IF-FAP to evaluate the robustness of detectors will be shown as follows.

In general CFAR detection problem, the signal under test $x(t)$ is usually obtained at the output of a square or linear law detector, and is simply described as

$$x_i(t) = i * s(t) + n(t) \quad i = 0, 1 \tag{6}$$

For $i = 0$, only the noise $n(t)$ is assumed to be present and this is actually the background clutter. For $i = 1$, both the signal $s(t)$ and the clutter are assumed to be present. A CFAR detector always gives the same probability of false alarm P_{fa} which can be pre-defined using the proper value for the threshold T. Assuming that the test statistics $\{z_k; k = 1, 2, \cdots\}$ are obtained through a square law detector, the value of T and the P_{fa} are related as follows:

$$P_{fa} = P(z > T\hat{\mu}) = \int_R \int_{T\hat{\mu}}^{\infty} dF(z) dG(\hat{\mu}) \tag{7}$$

where $F(z)$ is the cumulative distribution function (cdf) of clutter samples, $\hat{\mu}$ is an estimation of clutter power μ_0, and the distribution of statistic $\hat{\mu}$ is given by $G(\hat{\mu})$ which depends on the $F(z)$ and the structure of the estimator.

Substitute (7) to (5), the result of IF-FAP of CA-CFAR with n reference cells is given as follows:

$$\begin{aligned} IF_{FAP}(x; F) &= \int_R \int_{T\hat{\mu}_n}^{\infty} d(-F + \Delta_x)(z) dG(\hat{\mu}_n) \\ &+ \int_R \int_{T_n\hat{\mu}_n}^{\infty} dF(z) d\frac{\partial G(\hat{\mu}_n)}{\partial t} \\ &= -P_{fa} + G(x/T_n) + \int_R \int_{T_n\hat{\mu}_n}^{\infty} dF(z) d\frac{\partial G(\hat{\mu}_n)}{\partial t} \end{aligned} \tag{8}$$

where $\partial G(\hat{\mu}_n)/\partial t$ denotes the effect on the $G(\hat{\mu}_n)$ variety of the distribution contamination mixing with F. Then the first two items of the above equation denote the FAP variety rate due to the distribution contamination of the *test cell*, while the last item denotes that of the *reference cells*.

These CFAR detectors converge to the optimum detector when $n \to \infty$, where the clutter power is already known as $\mu_0 = \hat{\mu}(F)$, and $G(z) = \Delta_{\mu_0}(z)$, so

$$\begin{aligned} IF_{FAP}(x; F) &= \frac{\partial}{\partial t}[\int_{T_0\hat{\mu}(H)}^{\infty} dH|_{H=(1-t)F+t\Delta_x}] \\ &= \int_{T_0\mu_0}^{\infty} d(-F + \Delta_x) - T_0 \cdot IF(x; \hat{\mu}, F) f_0(T_0\mu_0) \\ &= -P_{fa} + \Delta_{T_0\mu_0}(x) - T_0 \cdot IF(x; \hat{\mu}, F) f_0(T_0\mu_0) \end{aligned} \tag{9}$$

where f_0 is the probability density function corresponding to F. It can be proven that (8) converges to (9) at every $x \in \mathbb{R}$ (but not uniform convergence). Moreover, all items in (9) are invariant along with the detector structure except for $IF(x; \hat{\mu}, F)$. Thus, we focus on the properties of $IF(x; \hat{\mu}, F)$ instead of IF-FAP itself and use it for describing the robustness of a group of CFAR detectors with the same structure and any number of reference cells. When P_{fa} and f_0 are changed to P_d and f_1 (f_1 denotes the probability density of clutter plus target), (9) turns to be the IF of detection probability, so the robustness of IF of detection probability can also be represented by $IF(x; \hat{\mu}, F)$ and needs no special discussion. After that, the gross-error sensitivity of $IF(x; \hat{\mu}, F)$ given by (2) can be used to evaluate the robustness of CFAR detectors.

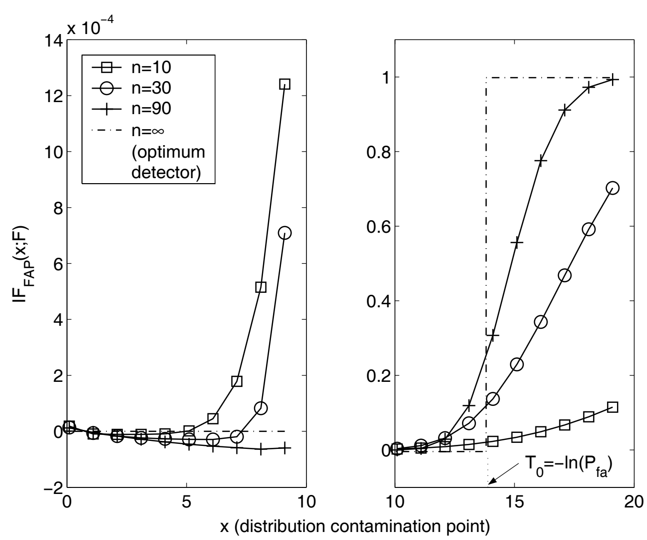

Fig. 1. The IF-FAP of CA-CFAR detectors with different number of reference cells.

As an example, the IF-FAP of CA-CFAR detector with difference number ($n = 10, 30, 90, \infty$) of reference cells are shown in Fig. 1, where it is supposed that the output of the square law detector obeys the exponential distribution, i.e., $F(z) = 1 - e^{-z/\mu_0}$, and $\mu_0 = 1, P_{fa} = 10^{-6}$.

2.2. Robustness Measures of Detectors Based on OS

From the result above, we mainly focus on the robustness of estimator $\hat{\mu}_n$ which can be asymptotically replaced by functional $\hat{\mu}(F)$. Then, equation (9) is also applicable for those CFAR detectors based on order statistics.

Consider those clutter power estimates in the CFAR detectors based on order statistics, $\{\hat{\mu}_n; n \geq 1\}$ are named L-estimators, which are of the form

$$\hat{\mu}_n(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} a_i X_{i:n}, \qquad (10)$$

where $X_{1:n}, \cdots, X_{n:n}$ are the ordered samples and the a_i are coefficients. A natural sequence of estimators is obtained if the weight a_i are generated by $a_i = \int_{[(i-1)/n, i/n]} h(u)du$, where $h : [0,1] \to \mathbb{R}$ satisfies $\int_{[0,1]} h(u)du > 0$. Then the corresponding functional is

$$\hat{\mu}(K) = \frac{\int x h(K(x))dK(x)}{c \int h(F(x))dF(x)}, \qquad (11)$$

It is Fisher consistent because of the standardization, where c is a multiplier ensuring $\hat{\mu}(F) = \mu_0$, and its IF is

$$IF(x; \hat{\mu}, F) = \frac{\int_{[0,x]} h(F(y))dy - \int [\int_{[0,r]} h(F(y))dy] dF(r)}{c \int h(F(y))dF(y)}. \qquad (12)$$

The IF of all the CFAR estimators based on order statistics can be achieved by (12), and three typical examples are given below where we assume that the underlying distribution $F(x) = 1 - e^{-x/\mu_0}$:

• When $h = \delta(\alpha), 0 < \alpha < 1$, the detector corresponds to the OS-CFAR [1]. So $c = F^{-1}(\alpha)/\mu_0 = -\ln(1-\alpha)$. Its IF and gross-error sensitivity respectively are

$$IF(x; \hat{\mu}, F) = -\frac{1}{\alpha' \ln \alpha'} [\frac{1 + \mathrm{sgn}(x + \ln \alpha')}{2} - \alpha'], \qquad (13)$$

$$\gamma^*(\alpha) = \begin{cases} -1/\ln \alpha' & \alpha \leq 1/2 \\ -\alpha/(\alpha' \ln \alpha') & \alpha > 1/2 \end{cases},$$

where $\alpha' = 1 - \alpha$. Then the *most B-robust* OS-CFAR detector is achieved at $\alpha = 1/2$ which minimize γ^* to $1/\ln 2$, the corresponding estimator is the median $\hat{\mu}_n = X_{[n/2]:n}$.

• When $h = 1_{[\alpha, 1-\beta]}, 0 \leq \alpha < 1 - \beta \leq 1$, the detector corresponds to the TM-CFAR [2]. So $c = [\beta(\ln \beta - 1) - \alpha'(\ln \alpha' - 1)]/[\alpha' - \beta]$. Its IF and gross-error sensitivity respectively are

$$IF(x; \hat{\mu}, F) = \frac{1}{c}(\frac{A}{\alpha' - \beta} - 1), \qquad (14)$$

where

$$A = \begin{cases} 0 & x < F^{-1}(x) \\ x + \ln \alpha' & F^{-1}(\alpha) \leq x < F^{-1}(1-\beta) \\ -\ln \beta + \ln \alpha' & x \geq F^{-1}(1-\beta) \end{cases},$$

and $\gamma^*(\alpha, \beta) = \frac{1}{c} \max\{1, \frac{\ln \alpha' - \ln \beta}{\alpha' - \beta} - 1\}$.

• When $h = 1_{[0, 1-\beta]}, 0 \leq \beta < 1$, the detector corresponds to the CMLD-CFAR [2], which is a special case of the TM-CFAR in which let $\alpha = 0$. Then its gross-error sensitivity is

$$\gamma^*(\beta) = \frac{1 - \beta}{\beta(\ln \beta - 1) - 1} \max\{1, \frac{-\ln \beta}{1 - \beta} - 1\}.$$

Then the *most B-robust* CMLD-CFAR detector is achieved at $\frac{-\ln \beta}{1-\beta} - 1 = 1$, and the solution value is denoted as $\beta^*(1 - \beta^* \approx 0.797)$. The corresponding estimator is $\hat{\mu}_n = \sum_{i=1}^{(1-\beta^*)n} X_{i:n}$.

3. DETECTION PERFORMANCE EVALUATED BY ASYMPTOTIC VARIANCE

As shown in the introduction, $V(\hat{\mu}, F)$ is the statistic measuring the asymptotical efficiency of $\{\hat{\mu_n}; n \geq 1\}$, that satisfies the Cramer-Rao inequality

$$V(\hat{\mu}, F) \geq \frac{1}{J(F)} = \frac{1}{\int (\frac{\partial \ln f(x,\mu)}{\partial \mu}|_{\mu_0})^2 d(F)}, \qquad (15)$$

where $J(F)$ is the *Fisher information*. The equality holds at the maximum likelihood estimator (MLE) (e.g. the mean estimation in CA-CFAR detector is a MLE at the exponential distribution), which is optimal in terms of detection performance. Thus, $V(\hat{\mu}, F)$ represents the efficiency of $\{\hat{\mu_n}; n \geq 1\}$ when estimating μ_0. Higher the efficiency is, closer the estimator is to the MLE, and less SNR loss it brings to the detector. Though this is an asymptotic property, we find that it brings pretty good results at finite n in the following examples.

Then, it's found that there is some close relationship between $V(\hat{\mu}, F)$ and detection performance which can be measured by the *averaging decision threshold* (ADT) presented by Rohling [1]. For it can hardly be analyzed mathematically, this relationship between two measures of the OS-CFAR detector as an example is illustrated in Fig. 2, where $V_{OS}(k, n)$ is achieved by substituting (13) into (3) when $\alpha = k/n$, and $V_{TM}(k, n)$ is achieved by substituting (14) into (3) when $\alpha = (k-1)/n, 1 - \beta = k/n$. It's clearly shown that these two kinds of asymptotic variances have the similar trend as the ADT when k takes the value form 1 to n, where the V_{TM} is more exact. The square root of ADT is used in the figure for convenience of comparison since only the relative values are concerned.

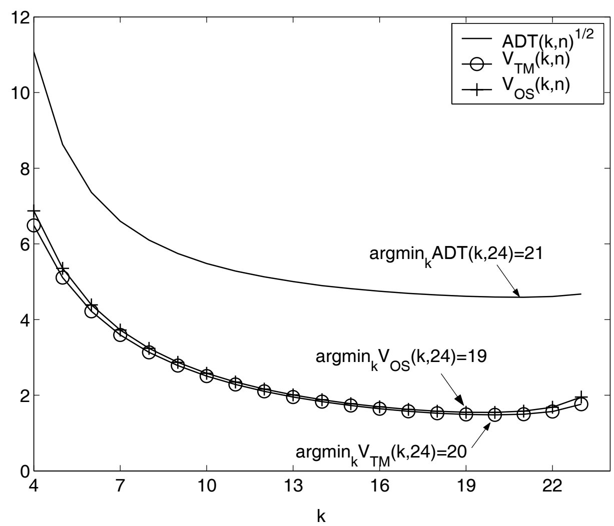

Fig. 2. The comparison of asymptotic variance and ADT, given $n = 24, P_{fa} = 10^{-6}$.

Table 1. The trend of the ratio of k^* minimizing $ADT(k,n)$ by n when $n \to \infty$, referring to the α^* minimizing $V_{OS}(\alpha)$, $P_{fa} = 10^{-6}$.

n	k^*	$\lceil \alpha^* n \rceil$	$ADT_{(k^*,n)}$	$ADT_{(\lceil \alpha^* n \rceil,n)}$	k^*/n
25	22	20	20.7	20.9	0.880
50	42	40	17.0	17.1	0.840
100	81	80	15.3	15.3	0.810
200	161	160	14.6	14.6	0.805
$\alpha^* = \arg\min_\alpha V_{OS}(\alpha)$					0.797

Moreover, an estimator minimizing $V(\hat{\mu}, F)$ is called *most efficient*. The most efficient OS-CFAR detector is achieved by substituting (13) to (3), it's followed that

$$V(\hat{\mu}, F) = \frac{\alpha}{(1-\alpha)\ln^2(1-\alpha)} \triangleq V_{OS}(\alpha). \quad (16)$$

Then let $\alpha^* = \arg\min_\alpha V_{OS}(\alpha)$, it has the same value as $1-\beta^*$ in Sect. 2.2. Meanwhile, the OS-CFAR detector which requests the smallest detection SNR is determined by the minimum of ADT lying at $k^* = \arg\min_k ADT(k,n)$. The comparison between k^*/n of different n and α^* are shown in Table 1. It can be seen that k^* and $\lceil \alpha^* n \rceil$ are very close, also $ADT(k^*, n)$ and $ADT(\lceil \alpha^* n \rceil, n)$ are almost the same, especially for large n. Then it illustrates that the one with the estimator $\hat{\mu}_n = X_{\lceil \alpha^* n \rceil : n}$ achieves the best detection performance approximately among all OS-CFAR detectors with n reference cells.

All the results show that $V(\hat{\mu}, F)$ is a new approximate measure of CFAR detection performance, and it has an explicit expression, which has more feasible computability than the ADT in many detectors in common use. The shortage of using the asymptotic variance is that it is a asymptotical result, and thus is naturally valid when large number of adjacent cells are referenced.

4. CONCLUSION

The robustness of a CFAR detector is described by the variation of FAP and detection probability due to the clutter distribution contamination, and it is mainly determined by the IF of clutter power estimator which yields the detection threshold. On the other hand, the efficiency of clutter power estimator is shown to be an approximate measure of the detection performance, which is represented by the asymptotic variance of that estimator. Moreover, the asymptotic variance has a fixed functional relationship with the IF.

It should be emphasized that there is always a tradeoff between these two features, because for adapting more situations a robust statistic must have some efficiency loss. That is also applicable to the CFAR detectors. Thus, we can construct high performance robust CFAR detectors by finding the most efficient estimator under the constraint of some robustness indexes (e.g. the optimal B-robust estimator). This work is under further research.

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