# HOW TO COMPLETE PARAUNITARY FILTER BANKS AND SIMULTANEOUSLY PRESERVE LINEAR PHASE?

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## ABSTRACT

In this paper, the completion of linear-phase paraunitary filter banks (LPPUFBs) is presented. Given an $M$-band LPPUFB-admissible scaling filter $H_0(z)$, the proposed method finds a *complete* parameterization such that the resulting $M-1$ bandpass/highpass filters $H_i(z)$ and the given filter $H_0(z)$ form a LPPUFB. Results from completion of a general PUFB will be used in combination with the LP-generating dyadic-based structure, so that the scaling filter $H_0(z)$ of the resulting LPPUFB is as prescribed and linear phase of the filter bank is guaranteed. The design procedure is demonstrated by an example.

## 1. INTRODUCTION

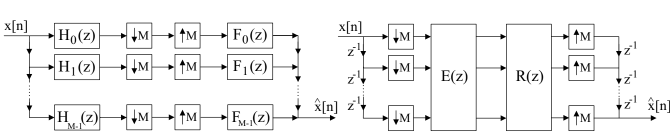
*Notations:* Bold-faced characters denote either a column vector or a matrix. For $i = 0, \ldots, M-1$, $\mathbf{e}_i$ is the $i$th unit vector of $\mathbb{R}^M$. $\mathbf{1}_M$ and $\mathbf{0}_M$ are the $M$-vectors of all ones and all zeros, respectively, and $\mathbf{I}_M$ and $\mathbf{J}_M$ denote the $M \times M$ identity and reverse identity matrices, respectively. An $m \times n$ constant matrix $\mathbf{A}$ is said to be unitary if $\mathbf{A}^\dagger \mathbf{A} = \mathbf{I}_n$. The McMillan degree and the order of $\mathbf{E}(z)$ are denoted by $\deg(\mathbf{E}(z))$ and $\mathrm{ord}(\mathbf{E}(z))$, respectively.

$M$-channel maximally decimated filter banks have recently found several applications in signal processing, data compression, and smooth approximation, etc. [1–5]. As shown in Fig. 1, for $i = 0, \ldots, M-1$, let $H_i(z)$ and $F_i(z)$ denote the analysis and synthesis filters, respectively, where the low-pass filters $H_0(z)$ and $F_0(z)$ are also referred to as the *scaling filters*, as they govern the $M$-band dilation equations for the underlying multiresolution analysis (MRA) of the Hilbert space. These filters are related to the *polyphase* representation through

$$\left[\, H_0(z) \;\; \ldots \;\; H_{M-1}(z) \,\right] = \left[\, 1 \;\; z^{-1} \;\; \ldots \;\; z^{1-M} \,\right] \mathbf{E}^T(z^M)$$

$$\left[\, F_0(z) \;\; \ldots \;\; F_{M-1}(z) \,\right] = \left[\, z^{1-M} \;\; \ldots \;\; z^{-1} \;\; 1 \,\right] \mathbf{R}(z^M)$$

where $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the type-I and type-II polyphase matrices, respectively. Perfect reconstruction (PR) requires that $\mathbf{E}(z)$ be non-singular for all $z$, so that the analysis filters $H_i(z)$ can be jointly inverted by the synthesis ones $F_i(z)$. Paraunitary filter banks (PUFB) are an important class for which $\mathbf{E}(z)$ is unitary on the unit circle; as a result, signal energy is preserved, and $F_i(z)$ can be found from $H_i(z)$ by inspection. If all the $H_i(z)$ have symmetry/antisymmetry in their impulse responses, the resulting filter bank is termed *linear-phase* [6, 7].




(a) $M$-channel filter bank (b) the polyphase representation

**Fig. 1**. $M$-band perfect reconstruction filter bank.

For a causal polyphase matrix $\mathbf{E}(z)$, the *McMillan degree* and the *order* of $\mathbf{E}(z)$ are two important concepts: the McMillan degree refers to the minimum number of delay elements required to implement $\mathbf{E}(z)$. A *minimal* structure is one which uses this minimum number of delay elements in it; as a contrast, the *order* of $\mathbf{E}(z)$ refers to the highest power of $z^{-1}$ appearing in $\mathbf{E}(z)$. For an $M$-band $\mathbf{E}(z)$ of order $L$, the filters $H_i(z)$ will in general have lengths $M(L+1)$. However, a PUFB of length $M(L+1)$ can have degree ranging from $L$ to $ML$. Gao *et al.* have recently proposed the *order-one* factorization for designing PUFBs with length constraint [8]. In particular, the following dyadic-based structure $\mathbf{W}_m(z)$ is involved:

**Lemma 1 (Order-One Factorization)** *The dyadic-based structure with parameter matrix* $\mathbf{w}_m$

$$\mathbf{W}_m(z) = \mathbf{I} - \mathbf{w}_m \mathbf{w}_m^\dagger + z^{-1} \mathbf{w}_m \mathbf{w}_m^\dagger, \quad \mathbf{w}_m^\dagger \mathbf{w}_m = \mathbf{I}_{\gamma_m} \quad (1.1)$$

*is the order-one paraunitary building block for some integer* $\gamma_m$ *with* $1 \le \gamma_m \le M$. *Any order-*$L$ *paraunitary polyphase matrix* $\mathbf{E}(z)$ *can be factored as*

$$\mathbf{E}(z) = \mathbf{W}_L(z)\, \mathbf{W}_{L-1}(z) \ldots \mathbf{W}_1(z)\, \mathbf{E}_0 \quad (1.2)$$

*for some* $M \times M$ *unitary* $\mathbf{E}_0$ *and some integers* $\gamma_1, \ldots, \gamma_L$. *This structure is referred to as the order-one factorization of* $\mathbf{E}(z)$. *It is complete for any given order* $L$.

We will demonstrate how to factor a polyphase *vector* into a product of $\mathbf{W}_m(z)$, based on which the completion of PUFB with linear phase property is obtained.

Filter bank completion is to address the following issue: given partial information in terms of the (admissible) scaling filter $H_0(z)$ of a perfect reconstruction filter bank, how does one come up with a representation that characterizes *all* possible solutions? The significance lies in that the best possible solution is ensured out of the structure used. Conventionally, two methods based on the so-called *degree-one factorization* have been proposed: the one in [9] serves as a way to reduce the number of free parameters and to obtain PUFBs with good performance; while the one in [10–12] requires further specification of an initial unitary matrix $\mathbf{E}_0$ for the method to work, but it is not clear how to choose $\mathbf{E}_0$ *a priori*. In both cases, the McMillan degree of the resulting PUFB is limited by that of the polyphase vector of $H_0(z)$, and thus the optimal performance of the PUFB given the filter $H_0(z)$ may not be obtained. A method has been proposed in [13] to overcome such degree limitation. This paper further elaborates on how to ensure linear phase in completing the PUFB.

## 2. PARAUNITARY FILTER BANK COMPLETION

### 2.1. General Theory

Though focusing on real-valued vectors, the following can be extended to a complex vector space in a straightforward fashion [14].

**Definition 1 (Householder Matrix)** *Given a vector $\mathbf{x} \in \mathbb{R}^M$, we define $\mathbf{R}[\mathbf{x}]$ to be the Householder matrix such that $\mathbf{R}[\mathbf{x}]\,\mathbf{x} = \|\mathbf{x}\|\mathbf{e}_0$ or that $\mathbf{x} = \|\mathbf{x}\|\,\mathbf{R}[\mathbf{x}]\,\mathbf{e}_0$. In case where $\mathbf{x} = \alpha\mathbf{e}_0$ for some scalar $\alpha > 0$, we define $\mathbf{R}[\alpha\mathbf{e}_0] = \mathbf{I}$.*

#### *2.1.1. Degree-One Completion of PUFBs*

As a special case of $\mathbf{W}_m(z)$ as in (1.1) with $\gamma_m = 1$, the following dyadic-based structure

$$\mathbf{V}_m(z) \triangleq \mathbf{I} - \mathbf{v}_m\mathbf{v}_m^\dagger + z^{-1}\mathbf{v}_m\mathbf{v}_m^\dagger \tag{2.1}$$

with $\|\mathbf{v}_m\| = 1$ is the *degree-one* paraunitary building block [9, 14]: $\deg(\mathbf{V}_m(z)) = 1$. Any degree-$N$ FIR $M \times 1$ causal lossless transfer function $\mathbf{p}(z)$ can be *uniquely* factorized as [9, 14]

$$\mathbf{p}(z) = \mathbf{V}_N(z)\ldots\mathbf{V}_1(z)\,\mathbf{p}_0 \tag{2.2}$$

for some degree-one building blocks $\mathbf{V}_m(z)$ as in (2.1) and $M\times 1$ unitary $\mathbf{p}_0 = \mathbf{p}(1)$. One can then parameterize any degree-$N$ PUFB $\mathbf{E}(z)$ with $\mathbf{p}(z)$ being the polyphase vector of the scaling filter $H_0(z)$ as [9, 13]:

$$\mathbf{E}^T(z) = \|\mathbf{p}_0\|\,\mathbf{V}_N(z)\ldots\mathbf{V}_1(z)\mathbf{R}[\mathbf{p}_0]\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{\Theta} \end{bmatrix}. \tag{2.3}$$

We refer to (2.3) as the *degree-one completion* of $\mathbf{E}(z)$, for which the $(M-1)\times(M-1)$ unitary matrix $\mathbf{\Theta}$ contains all the degrees of freedom, and can be parameterized by Householder matrices or by planar rotations [14, 15]. The resulting $\mathbf{E}(z)$ is paraunitary.

However, the degree-one completion of $\mathbf{E}(z)$ in (2.3) is *degree-constrained* in that it spans *only* the set of PUFBs of the same degree as $\mathbf{p}(z)$, with $\mathbf{p}^T(z)$ being the 0th row of $\mathbf{E}(z)$. Namely, the resulting $\mathbf{E}(z)$ is so constrained that

$$\deg(\mathbf{E}(z)) = \mathrm{ord}(\mathbf{E}(z)) \equiv \deg(\mathbf{p}(z)).$$

In general, $\deg(\mathbf{E}(z))$ is no less than $\mathrm{ord}(\mathbf{E}(z))$, and oftentimes it is desirable to allow for such a possibility. For example, suppose that $\mathbf{p}(z)$ of order $L$ is causal and is the vector of the $M$ polyphase components of a PUFB-admissible linear-phase filter $H_0(z)$. If $\mathbf{E}(z)$ of order $L$ corresponds to a linear-phase PUFB, its degree must be $\frac{ML}{2}$ [6, 14], which is greater than $L$ and hence is not possible under (2.3). Furthermore, experiments indicate that, given the order of $\mathbf{E}(z)$, PUFBs with better performance are usually obtained by allowing $\deg(\mathbf{E}(z)) > \mathrm{ord}(\mathbf{E}(z))$.

#### *2.1.2. Order-One Completion of PUFBs*

To relax the degree constraint on the resulting PUFBs which is intrinsic in the degree-one completion (2.3), the so-called *order-one completion* has been proposed in [13], which states that given an $M \times 1$ FIR lossless causal transfer function

$$\mathbf{p}(z) = \mathbf{a}_0 + \mathbf{a}_1 z^{-1} + \ldots + \mathbf{a}_L z^{-L}$$

of order or degree[1] $L$, there exist (non-unique) order-one PU building blocks $\hat{\mathbf{W}}_m(z)$ as in (1.1) such that[2]

$$\mathbf{p}(z) = \hat{\mathbf{W}}_L(z)\ldots\hat{\mathbf{W}}_1(z)\,\mathbf{p}_0 \tag{2.4}$$

where $\mathbf{p}_0 = \mathbf{p}(1)$. In particular, we have for $\hat{\mathbf{W}}_L(z)$ that its $M\times\gamma_L$ unitary parameter matrix $\hat{\mathbf{w}}_L$ takes the form

$$\hat{\mathbf{w}}_L \triangleq \left[\begin{array}{c|c} \Big| & \\ \frac{\mathbf{a}_L}{\|\mathbf{a}_L\|} & \mathbf{B}_L \\ \Big| & \end{array}\right] \tag{2.5}$$

where $\mathbf{B}_L$ consists of $\gamma_L - 1$ orthonormal columns that are orthogonal to both $\mathbf{a}_0$ and $\mathbf{a}_L$, and can be parameterized as follows:

$$\mathbf{B}_L = \mathbf{R}_{0L}\left[\begin{array}{c} -\ \mathbf{0}^T - \\ -\ \mathbf{0}^T - \\ \hline \mathbf{\Theta}_L \end{array}\right], \tag{2.6}$$

where $\mathbf{\Theta}_L$ is $(M-2)\times(\gamma_L-1)$ unitary and $\mathbf{R}_{0L}$ is the Householder transform such that $\begin{bmatrix} \frac{\mathbf{a}_0}{\|\mathbf{a}_0\|} & \frac{\mathbf{a}_L}{\|\mathbf{a}_L\|}\end{bmatrix} = \mathbf{R}_{0L}\begin{bmatrix}\mathbf{e}_0 & \mathbf{e}_1\end{bmatrix}$. The other order-one building blocks $\hat{\mathbf{W}}_m(z)$, $m < L$, are obtained in the same way (through some unitary $\mathbf{B}_m$ in the fashion of (2.6)) after repeated order reductions on $\mathbf{p}(z)$ [13]. As the choice of $\mathbf{B}_m$ is not unique (unless $\gamma_m = 1$), so is the order-one factorization (2.4) of $\mathbf{p}(z)$. The corresponding extra degrees of freedom are captured by $\mathbf{\Theta}_m$ of the above Householder parameterization of $\mathbf{B}_m$, and are used in completion of $\mathbf{E}(z)$, which is given by

$$\mathbf{E}^T(z) = \|\mathbf{p}_0\|\,\hat{\mathbf{W}}_L(z)\ldots\hat{\mathbf{W}}_1(z)\mathbf{R}[\mathbf{p}_0]\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{\Theta} \end{bmatrix}, \tag{2.7}$$

where $\mathbf{\Theta}$ is $(M-1)\times(M-1)$ unitary. Worth noting is the identification

$$\mathbf{R}[\mathbf{p}_0]\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{\Theta} \end{bmatrix} \equiv \mathbf{E}_0^T \tag{2.8}$$

between (2.7) and (1.2), which results if we set $z = 1$ in both (2.7) and (1.2).

Notice that (2.7) is transposed, not in the standard form given by (1.2). Plugging (2.8) in and transposing (2.7), we have

$$\begin{aligned} \mathbf{E}(z) &= \|\mathbf{p}_0\|\,\mathbf{E}_0\hat{\mathbf{W}}_1^T(z)\ldots\hat{\mathbf{W}}_L^T(z) \\ &= \|\mathbf{p}_0\|\,\mathbf{W}_L(z)\ldots\mathbf{W}_1(z)\mathbf{E}_0, \end{aligned} \tag{2.9}$$

where the mappings are

$$\mathbf{W}_m(z) \triangleq \mathbf{E}_0\hat{\mathbf{W}}_{L-m+1}^T(z)\mathbf{E}_0^\dagger \quad \text{and} \tag{2.10a}$$

$$\mathbf{w}_m \triangleq \mathbf{E}_0\hat{\mathbf{w}}_{L-m+1}^*. \tag{2.10b}$$

We refer to (2.9) as the *order-one completion* of (causal) $\mathbf{E}(z)$ of order $L$, of which the free parameters are embedded in each $\mathbf{\Theta}_m$ of $\mathbf{B}_m$ as well as in $\mathbf{\Theta}$. This is a *complete* parameterization of $\mathbf{E}(z)$ having $\mathbf{p}^T(z)$ as its top row.

[1] For an $M\times 1$ FIR causal $\mathbf{p}(z)$, $\deg(\mathbf{p}(z)) = \mathrm{ord}(\mathbf{p}(z))$.

[2] We reserve the notation $\mathbf{W}_m(z)$ for order-one completion of PUFBs to avoid possible confusion.

## 2.2. PUFB Completion with Linear Phase Property

It is shown in [16] that the following specialization of the order-one factorization (1.2) corresponds to $M$-channel LPPUFBs ($M$ even): for $m = 1, 2, \ldots, L$, parameterize the $M \times \frac{M}{2}$ unitary matrix $\mathbf{w}_m$ as

$$\mathbf{w}_m = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{U}_m \\ \mathbf{V}_m \end{bmatrix} \tag{2.11a}$$

and the initial unitary matrix $\mathbf{E}_0$ as

$$\mathbf{E}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_0\mathbf{J} \\ \mathbf{V}_0 & -\mathbf{V}_0\mathbf{J} \end{bmatrix}, \tag{2.11b}$$

where $\mathbf{U}_i$ and $\mathbf{V}_i$ are $\frac{M}{2} \times \frac{M}{2}$ orthonormal.

To ensure linear phase property of the completed PUFB, both (2.11a) and (2.11b) must be satisfied on top of the order-one completion (2.9). As $\mathbf{p}(z)$ is the polyphase vector of a LPPUFB-admissible scaling filter $H_0(z)$, it can be shown that $\mathbf{p}_0 \triangleq \mathbf{p}(z)|_{z=1}$ always takes the following symmetric form:

$$\|\mathbf{p}_0\|^{-1}\mathbf{p}_0^T = \begin{bmatrix} \hat{\mathbf{u}}_0^T & \hat{\mathbf{u}}_0^T\mathbf{J} \end{bmatrix}, \quad \text{some } \hat{\mathbf{u}}_0 \text{ with } \|\hat{\mathbf{u}}_0\| = \frac{1}{\sqrt{2}},$$

which should be the 0th row of $\mathbf{E}_0$ in (2.11b). Therefore, it is necessary and sufficient that the matrix $\mathbf{U}_0$ of $\mathbf{E}_0$ satisfy

$$\mathbf{e}_0^T\mathbf{U}_0 = \sqrt{2}\hat{\mathbf{u}}_0^T \tag{2.12}$$

in order for the scaling filter of the resulting PUFB to be as prescribed.

Having constrained $\mathbf{E}_0$ by (2.12), we now proceed to parameterize $\mathbf{w}_m$ in such a way that the condition of order-one completion on $\hat{\mathbf{w}}_{L-m+1}$ in (2.5) holds. As the procedure of order-one completion calculates $\hat{\mathbf{w}}_L$ before the others, $\mathbf{w}_1$ is determined first by (2.10). In particular, assuming $\mathbf{E}_0$ (or $\mathbf{U}_0$ and $\mathbf{V}_0$) has been chosen, it must be true that

$$\begin{bmatrix} \mathbf{U}_1 \\ \mathbf{V}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_0\mathbf{J} \\ \mathbf{V}_0 & -\mathbf{V}_0\mathbf{J} \end{bmatrix} \underbrace{\left[\begin{array}{c|c} \frac{\mathbf{a}_L^0}{\|\mathbf{a}_L\|} & \mathbf{B}_L^0 \\ \hline \frac{\mathbf{a}_L^1}{\|\mathbf{a}_L\|} & \mathbf{B}_L^1 \end{array}\right]}_{\hat{\mathbf{w}}_L} \tag{2.13}$$

where $\hat{\mathbf{w}}_L$ has been partitioned into upper and lower $\frac{M}{2}$ rows as shown. As $\mathbf{a}_L$ in $\hat{\mathbf{w}}_L$ is given, the 0th columns of $\mathbf{U}_1$ and $\mathbf{V}_1$ are fixed:

$$\mathbf{U}_1\mathbf{e}_0 = \mathbf{U}_0\left(\frac{\mathbf{a}_L^0}{\|\mathbf{a}_L\|} + \frac{\mathbf{J}\mathbf{a}_L^1}{\|\mathbf{a}_L\|}\right) \tag{2.14a}$$

$$\mathbf{V}_1\mathbf{e}_0 = \mathbf{V}_0\left(\frac{\mathbf{a}_L^0}{\|\mathbf{a}_L\|} - \frac{\mathbf{J}\mathbf{a}_L^1}{\|\mathbf{a}_L\|}\right). \tag{2.14b}$$

It remains to ensure that columns of $\mathbf{B}_L$ are orthogonal to both $\mathbf{a}_L$ and $\mathbf{a}_0$, as is required by order-one completion of $\mathbf{E}(z)$ given $\mathbf{p}^T(z)$ as the 0th row. The key is to parameterize $\mathbf{U}_1$ and $\mathbf{V}_1$ appropriately. Observe that any unitary $\mathbf{U}_1$ and $\mathbf{V}_1$ satisfying (2.14) will guarantee $\mathbf{B}_L^T\mathbf{a}_L = \mathbf{0}$ as a result of (2.13). To ensure $\mathbf{B}_L^T\mathbf{a}_0 = \mathbf{0}$, consider augmenting $\hat{\mathbf{w}}_L$ with $\frac{\mathbf{a}_0}{\|\mathbf{a}_0\|}$ and premultiplying by $\mathbf{E}_0$:

$$\mathbf{E}_0 \left[\begin{array}{c|c} \hat{\mathbf{w}}_L & \begin{matrix} \frac{\mathbf{a}_0^0}{\|\mathbf{a}_0\|} \\ \frac{\mathbf{a}_0^1}{\|\mathbf{a}_0\|} \end{matrix} \end{array}\right] = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c} \mathbf{U}_1 & \mathbf{u}_1 \\ \mathbf{V}_1 & \mathbf{v}_1 \end{array}\right] \tag{2.15}$$

where the column vectors $\mathbf{u}_1$ and $\mathbf{v}_1$ are given by

$$\mathbf{u}_1 = \mathbf{U}_0\left(\frac{\mathbf{a}_0^0}{\|\mathbf{a}_0\|} + \frac{\mathbf{J}\mathbf{a}_0^1}{\|\mathbf{a}_0\|}\right) \tag{2.16a}$$

$$\mathbf{v}_1 = \mathbf{V}_0\left(\frac{\mathbf{a}_0^0}{\|\mathbf{a}_0\|} - \frac{\mathbf{J}\mathbf{a}_0^1}{\|\mathbf{a}_0\|}\right). \tag{2.16b}$$

As the LHS of (2.15) is unitary by construction, so is the RHS. Therefore,

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \frac{1}{2} \left[\begin{array}{c|c} \mathbf{U}_1 & \mathbf{u}_1 \\ \mathbf{V}_1 & \mathbf{v}_1 \end{array}\right]^T \left[\begin{array}{c|c} \mathbf{U}_1 & \mathbf{u}_1 \\ \mathbf{V}_1 & \mathbf{v}_1 \end{array}\right] \tag{2.17}$$

from which we have

$$\mathbf{U}_1^T\mathbf{u}_1 = -\mathbf{V}_1^T\mathbf{v}_1 \tag{2.18}$$

$$\|\mathbf{u}_1\|^2 + \|\mathbf{v}_1\|^2 = 2. \tag{2.19}$$

Since $\mathbf{U}_1$ and $\mathbf{V}_1$ are unitary, we conclude that $\mathbf{u}_1$ and $\mathbf{v}_1$ are unit-norm.

As the 0th columns of $\mathbf{U}_1$ and $\mathbf{V}_1$ are fixed as in (2.14), one can show that the top-most equation in (2.18) is automatically satisfied as a result of (2.14). In particular, it can be shown that

$$\mathbf{e}_0^T\mathbf{U}_1^T\mathbf{u}_1 = \frac{\mathbf{a}_L^T\mathbf{J}\mathbf{a}_0}{\|\mathbf{a}_L\|\|\mathbf{a}_0\|} = -\mathbf{e}_0^T\mathbf{V}_1^T\mathbf{v}_1.$$

Therefore, it remains to jointly parameterize $\mathbf{U}_1$ and $\mathbf{V}_1$ in such a way that $\mathbf{e}_k^T\mathbf{U}_1^T\mathbf{u}_1 = -\mathbf{e}_k^T\mathbf{V}_1^T\mathbf{v}_1$ for $k = 1, 2, \ldots, \frac{M}{2} - 1$. This can be achieved by first choosing $\mathbf{V}_1$ subject to (2.14), and then letting $\mathbf{U}_1$ depend on it (or vice versa).

### 2.2.1. *Joint Parameterization of* $\mathbf{U}_1$ *and* $\mathbf{V}_1$

Suppose $\mathbf{V}_1$ has been chosen subject to (2.14). Then the RHS of (2.18) is a fixed vector with unit norm. This takes away a few degrees of freedom from $\mathbf{U}_1$, and we are interested in identifying the remaining degrees of freedom in $\mathbf{U}_1$, for which the Householder matrix is found to be useful. According to (2.18), one can write

$$\mathbf{u}_1 = \mathbf{U}_1(-\mathbf{V}_1^T\mathbf{v}_1) \triangleq \mathbf{U}_1\mathbf{b}_1.$$

By assumption, $\mathbf{b}_1$ is known and unit-norm, and can be written as $\mathbf{b}_1 = \mathbf{R}[\mathbf{b}_1]\,\mathbf{e}_0$. Hence,

$$\mathbf{u}_1 = \mathbf{U}_1\mathbf{R}[\mathbf{b}_1]\,\mathbf{e}_0,$$

which implies that the 0th column of $\mathbf{U}_1\mathbf{R}[\mathbf{b}_1]$ is $\mathbf{u}_1$. Applying $\mathbf{R}[\mathbf{u}_1]$ to $\mathbf{U}_1\mathbf{R}[\mathbf{b}_1]$, we arrive at

$$\mathbf{R}[\mathbf{u}_1]\mathbf{U}_1\mathbf{R}[\mathbf{b}_1] = \left[\begin{array}{c|c} 1 & \\ \hline & \mathbf{\Phi}_1 \end{array}\right]$$

for some $(\frac{M}{2} - 1) \times (\frac{M}{2} - 1)$ unitary matrix $\mathbf{\Phi}_1$ which consists of all the remaining degrees of freedom for $\mathbf{U}_1$ given $\mathbf{V}_1$. Therefore, we have shown that $\mathbf{U}_1$ can be parameterized as

$$\mathbf{U}_1 = \mathbf{R}[\mathbf{u}_1] \cdot \text{diag}(1, \mathbf{\Phi}_1) \cdot \mathbf{R}[\mathbf{b}_1] \tag{2.20}$$

once $\mathbf{V}_1$ has been chosen subject to (2.14). Note that $\mathbf{U}_1$ and $\mathbf{V}_1$ so obtained guarantee that $\hat{\mathbf{w}}_L$ given by

$$\hat{\mathbf{w}}_L = \frac{1}{2} \begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_0\mathbf{J} \\ \mathbf{V}_0 & -\mathbf{V}_0\mathbf{J} \end{bmatrix}^T \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{V}_1 \end{bmatrix} \tag{2.21}$$

is orthonormal and orthogonal to $\mathbf{a}_0$ as required, with the 0th column being exactly $\frac{\mathbf{a}_L}{\|\mathbf{a}_L\|}$.

### 2.2.2. *Order Reduction*

So far, we have parameterized the initial unitary matrix $\mathbf{E}_0$ and the building block $\mathbf{W}_1(z)$ or $\hat{\mathbf{W}}_L(z)$ of the order-one completion of $\mathbf{E}(z)$ as in (2.9), while simultaneously imposing the linear phase property. The obtained $\hat{\mathbf{W}}_L(z)$ is then used to reduce the order of $\mathbf{p}(z)$ by one as described in [13]. In particular, $\mathbf{p}'(z) \triangleq \hat{\mathbf{W}}_L(z^{-1})\mathbf{p}(z)$ will be LPPUFB-admissible with order $L-1$, and the above procedure is repeated on

$$\mathbf{p}'(z) = \mathbf{a}_0' + \mathbf{a}_1' z^{-1} + \ldots + \mathbf{a}_{L-1}' z^{-(L-1)}$$

to obtain $\mathbf{U}_2$ and $\mathbf{V}_2$, and so on and so forth. Note that once determined in the initial iteration, $\mathbf{E}_0$ will be used throughout the completion of $\mathbf{E}(z)$. The is a *complete* parameterization of all LPPUFBs $\mathbf{E}(z)$ having $\mathbf{p}^T(z)$ as its 0th row.

## 3. DESIGN EXAMPLE

We demonstrate the proposed theory by completing an $8\times40$ LPPUFB given a LPPUFB-admissible scaling filter $H_0(z)$ which is shown in Fig. 2 along with the resulting completion. As a contrast, using the same admissible $H_0(z)$, the completion of a general PUFB is shown in Fig. 3 based on the order-one completion [13], for which $M = 8$, $L = 4$, and $\gamma_m = M/2 = 4$, $m = 1, 2, 3, 4$. It is obvious that the proposed PUFB completion does ensure linear phase while guaranteeing $H_0(z)$ as prescribed.

## 4. CONCLUSION

We have presented LPPUFB completion given an $M$-band LPPUFB-admissible scaling filter $H_0(z)$. A *complete* parameterization of all such PUFBs is proposed which ensures that the resulting $M-1$ bandpass/highpass filters $H_i(z)$ and the given filter $H_0(z)$ form a LPPUFB. Results from completion of a general PUFB are used in combination with the LP-generating dyadic-based structure, so that the scaling filter of the resulting LPPUFB is as prescribed and linear phase of the filter bank is guaranteed. A completion example is presented.

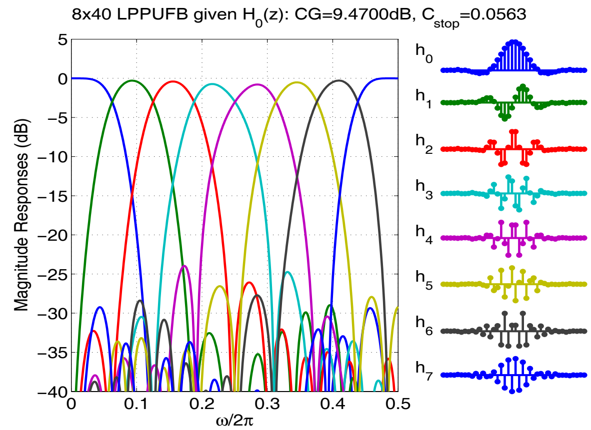



**Fig. 2**. Example of completing an $8\times40$ LPPUFB.

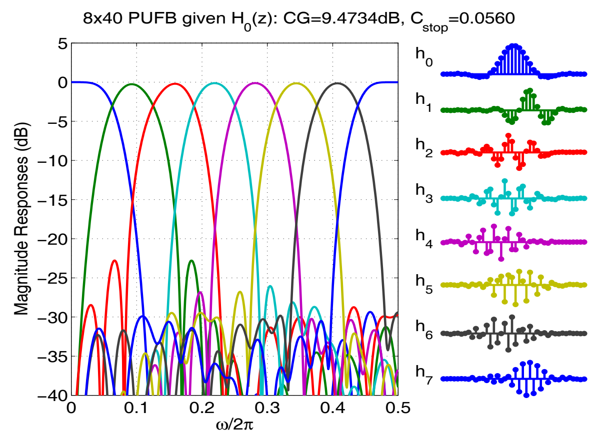



**Fig. 3**. Order-one completion of general $8\times40$ PUFB.

## 5. REFERENCES


[1] G. Strang and T. Nguyen, *Wavelets and Filter Banks*. Wellesley, MA: Wellesley-Cambridge Press, second ed., 1997.

[2] B. W. Suter, *Multirate and Wavelet Signal Processing*. San Diego: Academic Press, 1 ed., Jan. 1998.

[3] S. Mallat, *A Wavelet Tour of Signal Processing*. Academic Press, 1999.

[4] M. Vetterli and H. Kovačević, *Wavelets and Subband Coding*. Prentice Hall, 1995.

[5] C. Burrus, R. Gopinath, and H. Guo, *Introduction to Wavelets and Wavelet Transforms: A Primer*. Prentice Hall, 1998.

[6] R. de Queiroz, T. Nguyen, and K. Rao, "The GenLOT: Generalized linear-phase lapped orthogonal transform," *IEEE Trans. Signal Processing*, vol. 44, no. 3, pp. 497–507, 1996.

[7] T. Tran, R. de Queiroz, and T. Nguyen, "Linear-phase perfect reconstruction filter bank: lattice structure, design, and application in image coding," *IEEE Trans. Signal Processing*, vol. 48, pp. 133–147, Jan. 2000.

[8] X. Gao, T. Q. Nguyen, and G. Strang, "On factorization of $M$-channel paraunitary filterbanks," *IEEE Trans. Signal Processing*, vol. 49, pp. 1433–1446, Jul. 2001.

[9] P. Vaidyanathan, T. Nguyen, Z. Doğanata, and T. Saramäki, "Improved technique for design of perfect reconstruction FIR QMF banks with lossless polyphase matrices," *IEEE Trans. Signal Processing*, vol. 37, pp. 1042–1056, Jul. 1989.

[10] P. N. Heller and H. L. Resnikoff, "Regular $M$-band wavelets and applications," in *Proc. IEEE Int'l Conf. Acoust., Speech, Signal Processing*, vol. 3, pp. 229–232, Apr. 1993.

[11] P. N. Heller, "Rank $M$ wavelets with $N$ vanishing moments," *J. Matrix Anal. Appl.*, vol. 16, pp. 502–519, Apr. 1995.

[12] P. Steffen, P. N. Heller, R. A. Gopinath, and C. S. Burrus, "Theory of regular $M$-band wavelet bases," *IEEE Trans. Signal Processing*, vol. 41, pp. 3497–3510, Dec. 1993.

[13] Y.-J. Chen and K. Amaratunga, "On completion of $M$-channel perfect reconstruction filter banks with prescribed admissible FIR scaling filter," in *Proc. IEEE Int'l Symp. Circuits Syst.*, May 2004.

[14] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Prentice Hall, 1992.

[15] F. D. Murnaghan, *The Unitary and Rotation Groups*. Washington, D.C.: Spartan Books, 1962.

[16] K. Amaratunga, Y.-J. Chen, and S. Oraintara, "Characterization of regular linear-phase paraunitary filter banks using dyadic-based structures," in *Proc. IEEE Int'l Symp. Circuits Syst.*, May 2004.