DYADIC-BASED STRUCTURE FOR REGULAR BIORTHOGONAL FILTER BANKS WITH LINEAR PHASE

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ABSTRACT

The purpose of this paper is twofold: one is to establish a framework for general biorthogonal filter banks (BOFBs) with *structural* regularity; the other is to identify the connection between the general structure used here and the one commonly used for linear-phase biorthogonal filter banks (a.k.a. generalized lapped biorthogonal transform or GLBT). The latter also leads naturally to the same reduced number of free parameters as GLBT. We first revisit a *minimal* structure of BOFBs using order-one dyadic-based building blocks, by which BOFBs with length constraint can be designed. Conditions for filter bank regularity on the dyadic-based structure are derived and specialized to the case of GLBT. Design examples are presented.

1. INTRODUCTION

Notations: Bold-faced characters denote either a column vector or a matrix. For $i = 0, \ldots, M-1$, $\mathbf{e}_i$ is the ith unit vector of $\mathbb{C}^M$. $\mathbf{1}_M$ and $\mathbf{0}_M$ are the M-vectors of all ones and all zeros, respectively, and $\mathbf{I}_M$ and $\mathbf{J}_M$ denote the $M \times M$ identity and reverse identity matrices. $\rho(\mathbf{A})$ denotes the rank of $\mathbf{A}$.

Recently, M-channel filter banks have found several applications in signal processing [1–5]. Biorthogonal filter banks (BOFBs), in particular, have been employed as a transform coder in image compression application where their coding performances have shown to be a significant improvement over other traditional transforms [6, 7]. In addition to its frequency selectivity and coding gain, an optimized BOFB for the purpose of image coding usually has two other properties imposed: (i) linear phase (symmetry and anti-symmetry of the filters' impulse responses) and (ii) regularity. In [6], a modular structure for parameterizing BOFBs with linear phase is presented, in which linear phase and perfect reconstruction (PR) properties are structurally imposed. It is a modified version of that proposed for paraunitary filter banks (PUFBs) [8]. In [7], the structure is further extended in order to additionally impose regularity on the transform.

Regularity is fundamental to the filter bank theory and is closely related to the smoothness of the corresponding wavelet basis [1]. An M-channel filter bank is said to be (K_a, K_s)-*regular* if the analysis and synthesis lowpass filters $H_0(z)$ and $F_0(z)$ have a zero of multiplicity K_a and K_s, respectively, at the Mth roots of unity $e^{j2\pi m/M}$ for $m = 1, \ldots, M-1$, which is equivalent to

$$\frac{d^n}{dz^n}\Big\{\left[z^{1-M} \ \ldots \ z^{-1} \ 1\right]\mathbf{R}(z^M)\Big\}\Big|_{z=1} = \left[\, d_n \ 0 \ \ldots \ 0\,\right] \quad (1.1a)$$

$$\frac{d^m}{dz^m}\Big\{\left[1 \ z^{-1} \ \ldots \ z^{1-M}\right]\mathbf{E}^T(z^M)\Big\}\Big|_{z=1} = \left[\, c_m \ 0 \ \ldots \ 0\,\right] \quad (1.1b)$$

for some $c_m, d_n \neq 0$, $m = 0, 1, \ldots, K_s - 1$, $n = 0, 1, \ldots, K_a - 1$. This states that the multiplicity of zeros at DC of the analysis (synthesis) bandpass/highpass filters is equal to that of the synthesis (analysis) lowpass filter [9, 10]. Regular filter banks are desirable to many applications such as signal interpolation and data compression [1–5].

For a causal $M \times M$ polyphase matrix $\mathbf{E}(z)$, the *McMillan degree* and the *order* are two distinct but important concepts. The *(McMillan) degree* of $\mathbf{E}(z)$ refers to the minimum number of delay elements required for its implementation. A *minimal* structure of $\mathbf{E}(z)$ is one which uses this minimum number of delay elements in it; as a contrast, the *order* of $\mathbf{E}(z)$ refers to the highest power of z^{-1} in $\mathbf{E}(z)$. As a result, the degree is no less than the order.

In this paper, we consider the class of causal FIR M-band biorthogonal filter banks of order L spanned by

$$\mathbf{E}(z) = \mathbf{W}_L(z)\ldots\mathbf{W}_1(z)\mathbf{E}_0 \quad (1.2)$$

with an FIR inverse, where $\mathbf{E}_0$ is non-singular and each $\mathbf{W}_m(z)$ is the first-order biorthogonal (dyadic-based) building block given by

$$\mathbf{W}_m(z) = \mathbf{I} - \mathcal{U}_m\mathcal{V}_m^\dagger + z^{-1}\mathcal{U}_m\mathcal{V}_m^\dagger \quad (1.3)$$

where the $M \times \gamma_m$ parameter matrices $\mathcal{U}_m$ and $\mathcal{V}_m$ satisfy

$$\mathcal{V}_m^\dagger\mathcal{U}_m = \begin{bmatrix} 1 & \times & \times & \ldots & \times \\ 0 & 1 & \times & \ldots & \times \\ 0 & 0 & 1 & \ldots & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}_{\gamma_m \times \gamma_m} \triangleq \mathbf{\Delta}_m \quad (1.4)$$

for some integer $1 \leq \gamma_m \leq M$, where $\times$ indicates possibly nonzero elements. This is a generalization of the paraunitary order-one factorization given in [11] where $\mathcal{U}_m = \mathcal{V}_m$, and has been used for factoring the BOLT [12].

Remarks:

1. Since $\rho(\mathcal{V}_m^\dagger\mathcal{U}_m) = \gamma_m$, the McMillan degree of $\mathbf{W}_m(z)$ as in (1.3) is γ_m.
2. The construction in (1.2) completely spans all causal FIR BOFBs having FIR inverses, up to a factor unimodular in z^{-1} [12]. The spanned analysis filters have filter lengths no greater than $M(L+1)$, and the McMillan degree of $\mathbf{E}(z)$ ranges from L to ML where L is the order of the FB.
3. A causal Type-II synthesis polyphase matrix $\mathbf{R}(z)$ can be

$$\mathbf{R}(z) = z^{-L}\mathbf{E}_0^{-1}\mathbf{W}_1^{-1}(z)\ldots\mathbf{W}_L^{-1}(z). \quad (1.5)$$

As a result of the possibly nonzero off-diagonal elements in (1.4), the synthesis bank can have filter lengths different from $M(L+1)$. In fact, the lengths of the synthesis filters are bounded by $M(\mu+1)$ from above, where $\mu = \sum_{m=1}^{L} \gamma_m$ is the McMillan degree of $\mathbf{E}(z)$. The choice $\mathbf{\Delta}_m = \mathbf{I}_{\gamma_m}$ results in equal filter lengths for the analysis and synthesis banks.

2. REGULAR BIORTHOGONAL FILTER BANKS

2.1. $(1,1)$-Regular BOFBs

Regularity can be *structurally* imposed on the standard dyadic form (1.2). To demonstrate this point, we consider the design of $(1,1)$-regular BOFBs. In this case, it is true that $\mathbf{E}(z^M)\mathbf{e}_M(z)|_{z=1} = \mathbf{E}_0\mathbf{1}_M = c_0\mathbf{e}_0$ and $\mathbf{R}^T(z^M)\mathbf{J}\mathbf{e}_M(z)|_{z=1} = \mathbf{E}_0^{-T}\mathbf{1}_M = d_0\mathbf{e}_0$, where the delay chain $\mathbf{e}_M(z) = \begin{bmatrix} 1 & z^{-1} & \ldots & z^{1-M} \end{bmatrix}^T$. This implies that the entries of the top row of $\mathbf{E}_0$ (and $\mathbf{E}_0^{-T}$) be equal.

To parameterize such non-singular matrices having identical entries in the top row, a method is proposed in [7]. In particular, for any non-singular $\mathbf{E}_0$, one can write

$$\mathbf{E}_0 = \mathbf{RDLP} \tag{2.1}$$

where $\mathbf{R} = \mathbf{I} + \mathbf{e}_0 \begin{bmatrix} 0 & \mathbf{r}^T \end{bmatrix}$ and $\mathbf{L} = \mathbf{I} + \begin{bmatrix} 0 & \boldsymbol{\ell}^T \end{bmatrix}^T \mathbf{e}_0^T$ with $\mathbf{r} = \begin{bmatrix} r_1 & r_2 & \ldots & r_{M-1} \end{bmatrix}^T$ and $\boldsymbol{\ell} = \begin{bmatrix} \ell_1 & \ell_2 & \ldots & \ell_{M-1} \end{bmatrix}^T$, $\mathbf{D} = \begin{bmatrix} \alpha & \mathbf{0}^T \\ \mathbf{0} & \bar{\mathbf{E}}_0 \end{bmatrix}$ is non-singular, and $\mathbf{P}$ is obtained by exchanging the 0th row and some other one of $\mathbf{I}$. By construction, $\mathbf{R}$ and $\mathbf{L}$ correspond to lifting steps with lifting coefficients r_i and ℓ_i.

It can be shown that the one-regular conditions for the synthesis and analysis banks simplify to

$$K_s \geq 1 \Longrightarrow \begin{bmatrix} 1 & \ell_1{+}1 & \ldots & \ell_{M-1}{+}1 \end{bmatrix}^T = \frac{c_0}{\alpha}\mathbf{e}_0, \tag{2.2a}$$

$$K_a \geq 1 \Longrightarrow \mathbf{D}^{-T}\left(\mathbf{1}_M - \sum_{i=1}^{M-1} \ell_i \mathbf{e}_0\right) = d_0 \begin{bmatrix} 1 \\ \mathbf{r} \end{bmatrix}, \tag{2.2b}$$

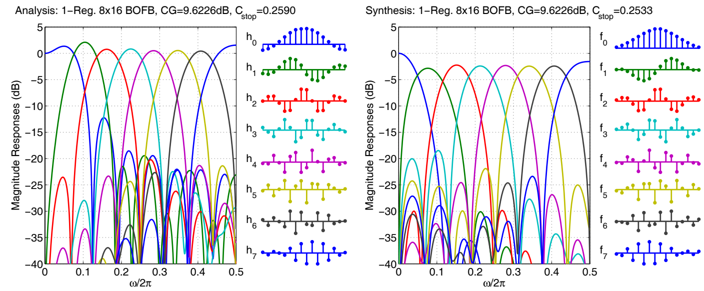
respectively. $(1,1)$-regular BOFBs are furnished by the following theorem, whose proof is left as an exercise.

Theorem 1 *M-band BOFBs as in (1.2) and (1.5) are $(1,1)$-regular if and only if*

$$\begin{aligned} \boldsymbol{\ell} &= -\mathbf{1}_{M-1}, \quad c_0 = \alpha, \quad c_0 d_0 = M, \quad \textit{and} \\ \mathbf{r} &= \frac{1}{d_0}\bar{\mathbf{E}}_0^{-T}\mathbf{1}_{M-1}, \end{aligned}$$

where $\mathbf{E}_0$ is parameterized as in (2.1).

Example 1: In this example, a $(1,1)$-regular, 8-channel, 16-tap BOFB is designed using the proposed theory. Related parameters are: $L = 1$, $\gamma_1 = 4$, and $\mathbf{\Delta}_1 = \mathbf{I}$ for simplicity. Each non-singular matrix is parameterized using the *QR* factorization [13]. Fig. 1 shows the resulting design with coding gain 9.6226dB. Note that BOFBs with better performance can be obtained if nonzero off-diagonal entries of $\mathbf{\Delta}_1$ are permitted as in (1.4).



(a) Analysis bank (b) Synthesis bank

Fig. 1. (1,1)-reg. 8×16 BOFB: impulse and frequency responses.

2.2. $(1,2)$-Regular BOFBs

Due to limited space, we summarize the result for two-regular synthesis bank below. That for analysis bank can be similarly derived and is skipped.

Theorem 2 *M-band BOFBs as in (1.2) and (1.5) are $(1,2)$-regular if and only if (2.2a) holds and*

$$-c_0 M \sum_{m=1}^{L} \mathbf{w}_m - \mathbf{E}_0\mathbf{b}_M = c_1\mathbf{e}_0, \quad c_1 \neq 0 \tag{2.3a}$$

where $\mathbf{w}_m = \mathcal{U}_m\mathcal{V}_m^{\dagger}\mathbf{e}_0$ and $\mathbf{b}_M = \begin{bmatrix} 0 & 1 & \ldots & M{-}1 \end{bmatrix}^T$. If $\mathbf{P}$ in (2.1) is chosen to be $\mathbf{I}$, Eqn. (2.3a) further simplifies to

$$\bar{\mathbf{E}}_0\bar{\mathbf{b}}_M = -c_0 M \sum_{m=1}^{L} \bar{\mathbf{w}}_m \tag{2.3b}$$

where $\mathbf{w}_m \triangleq \begin{bmatrix} w_0^m & \bar{\mathbf{w}}_M^T \end{bmatrix}^T$ and $\mathbf{b}_M \triangleq \begin{bmatrix} 0 & \bar{\mathbf{b}}_M^T \end{bmatrix}^T$.

Remarks: As $K_a = 1$, $\bar{\mathbf{E}}_0$ is not constrained in any way other than non-singularity. Hence in the case of (2.3b), assuming all $\mathbf{w}_m$ are known, $\bar{\mathbf{E}}_0$ can be parameterized similarly as in [7, Thm. 3] so that (2.3b) is satisfied.

Example 2: Using the above thoerem with (2.3b), we design a $(1,2)$-regular BOFB of eight channels ($M = 8$) and length 16 ($L = 1$). Again, we choose $\gamma_1 = 4$ and $\mathbf{\Delta}_1 = \mathbf{I}$ for simplicity. Fig. 2 shows the resulting design with coding gain 9.6031dB. Observe the double zeros of $F_0(z)$ at the aliasing frequencies, implying a two-regular synthesis bank. The synthesis basis is thus smoother than the analysis basis.

3. LINEAR-PHASE BIORTHOGONAL FILTER BANKS

An M-channel (M even) linear-phase biorthogonal filter bank (BOLP) of order L can be factored as follows [6, 8]

$$\mathbf{E}(z) = \mathbf{G}_L(z)\mathbf{G}_{L-1}(z)\ldots\mathbf{G}_1(z)\mathbf{E}_0^{LP} \tag{3.1}$$

where $\mathbf{G}_m(z) = \mathbf{\Gamma}_m\mathbf{W}\mathbf{\Lambda}(z)\mathbf{W}$ is the BOLP building block, and the initial non-singular matrix $\mathbf{E}_0^{LP} = \mathbf{\Gamma}_0\tilde{\mathbf{I}}\mathbf{W}\tilde{\mathbf{I}}$, with

$$\mathbf{\Gamma}_m = \begin{bmatrix} \mathbf{U}_m & \mathbf{0}_{M/2} \\ \mathbf{0}_{M/2} & \mathbf{V}_m \end{bmatrix}, \qquad \mathbf{W} = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{I}_{M/2} \\ \mathbf{I}_{M/2} & -\mathbf{I}_{M/2} \end{bmatrix},$$

$$\mathbf{\Lambda}(z) = \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{0}_{M/2} \\ \mathbf{0}_{M/2} & z^{-1}\mathbf{I}_{M/2} \end{bmatrix}, \quad \text{and} \quad \tilde{\mathbf{I}} = \begin{bmatrix} \mathbf{I}_{M/2} & \mathbf{0}_{M/2} \\ \mathbf{0}_{M/2} & \mathbf{J}_{M/2} \end{bmatrix}.$$

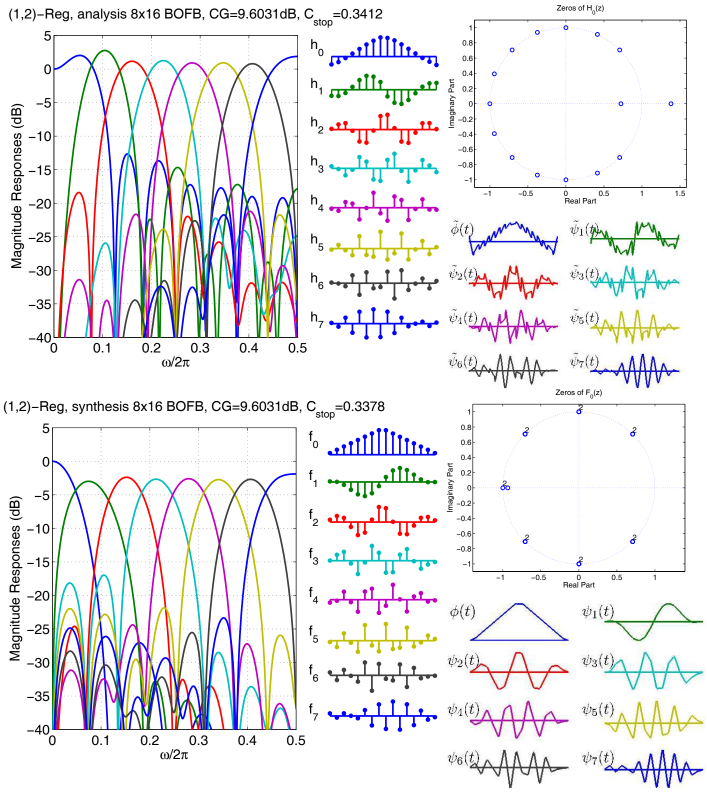


Fig. 2. (1,2)-reg. 8×16 BOFB: impulse and frequency responses, along with zero plots of $H_0(z)$ and $F_0(z)$, and the wavelet bases.

The $\mathbf{U}_m$ and $\mathbf{V}_m$ are $M/2\times M/2$ non-singular. Fig. 3 shows the lattice structure of an eight-channel BOLP of order L.

3.1. BOLP in Standard Dyadic Form (1.2)

By construction, each FIR LP building block $\mathbf{G}_m(z)$ is causal of order one and has an anticausal inverse. Namely, it is a BOLT [12], and it follows that one can *always* express $\mathbf{G}_m(z)$ in terms of the first-order BO building block $\mathbf{W}_m(z)$, with a suitable choice of parameter matrices $\mathcal{U}_m$ and $\mathcal{V}_m$ (this in fact is a rather deep result). In particular, one can show that (with subscripts $M/2$ dropped for simplicity)

$$\begin{aligned}\mathbf{G}_L(z) &= \mathbf{\Gamma}_L\left\{\mathbf{I}+\frac{(z^{-1}-1)}{2}\begin{bmatrix}\mathbf{I} & -\mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{bmatrix}\right\}\\ &= \left\{\mathbf{I}+\frac{(z^{-1}-1)}{2}\mathbf{\Gamma}_L\begin{bmatrix}\mathbf{I} & -\mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{bmatrix}\mathbf{\Gamma}_L^{-1}\right\}\mathbf{\Gamma}_L\\ &= \left\{\mathbf{I}+\frac{(z^{-1}-1)}{2}\begin{bmatrix}\mathbf{I} & -\mathbf{U}_L\mathbf{V}_L^{-1}\\ -\mathbf{V}_L\mathbf{U}_L^{-1} & \mathbf{I}\end{bmatrix}\right\}\mathbf{\Gamma}_L\\ &\triangleq \mathbf{W}_L(z)\mathbf{\Gamma}_L, \qquad (3.2)\end{aligned}$$

where the first-order BO building block $\mathbf{W}_L(z)$ is given by

$$\mathbf{W}_L(z)=\mathbf{I}+(z^{-1}-1)\frac{1}{2}\begin{bmatrix}\mathbf{I} & -\mathbf{U}_L\mathbf{V}_L^{-1}\\ -\mathbf{V}_L\mathbf{U}_L^{-1} & \mathbf{I}\end{bmatrix}.$$

The trailing factor $\mathbf{\Gamma}_L$ is absorbed by $\mathbf{G}_{L-1}(z)$ so that

$$\begin{aligned}\mathbf{G}_L(z)\mathbf{G}_{L-1}(z) &= \mathbf{W}_L(z)\begin{bmatrix}\mathbf{U}_L & \mathbf{0}\\ \mathbf{0} & \mathbf{V}_L\end{bmatrix}\mathbf{G}_{L-1}(z)\\ &= \mathbf{W}_L(z)\begin{bmatrix}\tilde{\mathbf{U}}_{L-1} & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{V}}_{L-1}\end{bmatrix}\mathbf{W}\mathbf{\Lambda}(z)\mathbf{W}\\ &= \mathbf{W}_L(z)\mathbf{W}_{L-1}(z)\begin{bmatrix}\tilde{\mathbf{U}}_{L-1} & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{V}}_{L-1}\end{bmatrix},\end{aligned}$$

where the relation in (3.2) has been employed in the last equality with $\tilde{\mathbf{U}}_m \triangleq \mathbf{U}_L\mathbf{U}_{L-1}\ldots\mathbf{U}_m$ and $\tilde{\mathbf{V}}_m \triangleq \mathbf{V}_L\mathbf{V}_{L-1}\ldots\mathbf{V}_m$, and $\mathbf{W}_m(z)$ is given by

$$\mathbf{W}_m(z)=\mathbf{I}+(z^{-1}-1)\frac{1}{2}\begin{bmatrix}\mathbf{I} & -\tilde{\mathbf{U}}_m\tilde{\mathbf{V}}_m^{-1}\\ -\tilde{\mathbf{V}}_m\tilde{\mathbf{U}}_m^{-1} & \mathbf{I}\end{bmatrix}. \qquad (3.3)$$

We can carry out the same procedure until arriving at

$$\mathbf{E}(z)=\mathbf{W}_L(z)\ldots\mathbf{W}_1(z)\underbrace{\begin{bmatrix}\tilde{\mathbf{U}}_0 & \mathbf{0}\\ \mathbf{0} & \tilde{\mathbf{V}}_0\end{bmatrix}\tilde{\mathbf{I}}\mathbf{W}\tilde{\mathbf{I}}}_{\mathbf{E}_0}, \qquad (3.4)$$

which is in the standard dyadic form (1.2).

3.2. LP-Generating Standard Dyadic Structure

Consider the first-order BO building block as in (3.3). The corresponding parameter matrices $\mathcal{U}_m$ and $\mathcal{V}_m$ can be chosen to be

$$\begin{aligned}\mathcal{U}_m &= \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{I}\\ -\tilde{\mathbf{V}}_m\tilde{\mathbf{U}}_m^{-1}\end{bmatrix}\mathbf{S}_m, \qquad (3.5a)\\ \mathcal{V}_m^{\dagger} &= \frac{\mathbf{S}_m^{-1}}{\sqrt{2}}\begin{bmatrix}\mathbf{I} & -(\tilde{\mathbf{V}}_m\tilde{\mathbf{U}}_m^{-1})^{-1}\end{bmatrix} \qquad (3.5b)\end{aligned}$$

for any $\gamma_m\times\gamma_m$ non-singular matrix $\mathbf{S}_m$. Note that for the LP case, $\gamma_m\equiv\frac{M}{2}$ and $\mathbf{\Delta}_m\equiv\mathbf{I}$ for all m. Along with the initial non-singular matrix $\mathbf{E}_0 = diag\{\tilde{\mathbf{U}}_0,\tilde{\mathbf{V}}_0\}\tilde{\mathbf{I}}\mathbf{W}\tilde{\mathbf{I}}$, the choice in (3.5) guarantees that the standard dyadic form (1.2) preserves the linear phase property.

3.3. Degrees of Freedom

The standard dyadic form (1.2) provides a new parameterization of BOLP by defining

$$\begin{aligned}&\hat{\mathbf{U}}_0=\tilde{\mathbf{U}}_0,\quad \hat{\mathbf{V}}_0=\tilde{\mathbf{V}}_0,\quad \text{and}\\ &\hat{\mathbf{V}}_m=-\tilde{\mathbf{V}}_m\tilde{\mathbf{U}}_m^{-1},\quad m=1,2,\ldots,L,\end{aligned}$$

and forming the parameter matrices according to

$$\mathcal{U}_m=\frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{I}\\ \hat{\mathbf{V}}_m\end{bmatrix},\quad \mathcal{V}_m^{\dagger}=\frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{I} & \hat{\mathbf{V}}_m^{-1}\end{bmatrix}. \qquad (3.6)$$

Namely, there are in total $L+2$ non-singular matrices $\hat{\mathbf{U}}_0$ and $\hat{\mathbf{V}}_i$ of size $M/2\times M/2$, consisting of free parameter. This is less than $2L+2$ as in (3.1) and is as efficient as the reduced-parameter structure for BOLPs established in [10,11]. Note that starting with a set of (original) parameter matrices $\mathbf{U}_m$ and $\mathbf{V}_m$ as in (3.1), one can always obtain a corresponding *smaller* set of matrices $\hat{\mathbf{U}}_0$ and $\hat{\mathbf{V}}_m$. Hence, the completeness of the structure is not affected by the proposed parameterization.

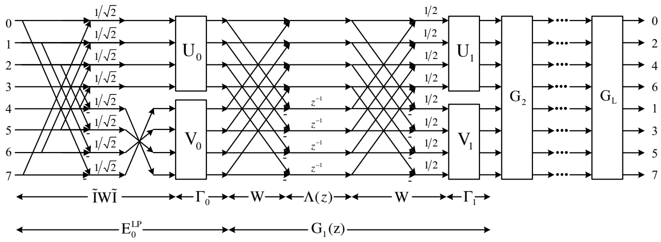


Fig. 3. Lattice structure for biorthogonal LP lapped transform.

Theorem 3 *The standard dyadic form (1.2) spans all M-band GLBTs (M even) if it is parameterized by non-singular matrices $\hat{\mathbf{U}}_0$ and $\hat{\mathbf{V}}_i$ of size $\frac{M}{2} \times \frac{M}{2}$ in such a way that*

$$\mathbf{E}_0 = diag\{\hat{\mathbf{U}}_0, \hat{\mathbf{V}}_0\}\,\tilde{\mathbf{I}}\mathbf{W}\tilde{\mathbf{I}} = \frac{1}{\sqrt{2}}\begin{bmatrix}\hat{\mathbf{U}}_0 & \hat{\mathbf{U}}_0\mathbf{J}\\ \hat{\mathbf{V}}_0\mathbf{J} & -\hat{\mathbf{V}}_0\end{bmatrix} \qquad (3.7)$$

and the parameter matrices $\mathcal{U}_m$ and $\mathcal{V}_m$ of $\mathbf{W}_m(z)$ are as given in (3.6) in terms of $\hat{\mathbf{V}}_m$.

4. REGULAR LINEAR-PHASE BIORTHOGONAL FILTER BANKS

As we can now parameterize any GLBT using the standard dyadic form (1.2), the regularity conditions on the general dyadic-based BO structure without the LP constraint can be applied. In particular, we will see how they simplify under the LP assumption.

Suppose $\mathbf{R}(z)$ is at least one-regular. It follows that $\mathbf{E}_0\mathbf{1}_M = c_0\mathbf{e}_0$ for some $c_0 \neq 0$. Substituting (3.7) gives

$$c_0\mathbf{e}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix}\hat{\mathbf{U}}_0 & \hat{\mathbf{U}}_0\mathbf{J}\\ \hat{\mathbf{V}}_0\mathbf{J} & -\hat{\mathbf{V}}_0\end{bmatrix}\begin{bmatrix}\mathbf{1}\\ \mathbf{1}\end{bmatrix} = \sqrt{2}\begin{bmatrix}\hat{\mathbf{U}}_0\mathbf{1}\\ \mathbf{0}\end{bmatrix},$$

or $\frac{c_0}{\sqrt{2}}\mathbf{e}_0 = \hat{\mathbf{U}}_0\mathbf{1}_{\frac{M}{2}}$, which is equivalent to the property that the elements of the top row of $\hat{\mathbf{U}}_0$ are equal. Similarly, if $\mathbf{E}(z)$ is at least one-regular, one arrives at $\frac{d_0}{\sqrt{2}}\mathbf{e}_0 = \hat{\mathbf{U}}_0^{-T}\mathbf{1}_{\frac{M}{2}}$. The technique employed in Sec. 2 applies here.

Now, suppose $\mathbf{R}(z)$ is at least two-regular. Plugging (3.6) into (2.3a) results in

$$-\frac{c_0 M}{2}\sum_{m=1}^{L}\begin{bmatrix}\mathbf{e}_0\\ \hat{\mathbf{V}}_m\mathbf{e}_0\end{bmatrix} - \mathbf{E}_0\mathbf{b}_M = c_1\begin{bmatrix}\mathbf{e}_0\\ \mathbf{0}\end{bmatrix} \qquad (4.1)$$

where $\mathbf{e}_0 \in \mathbb{R}^{M/2}$. Now using (3.7) and noting that $\mathbf{b}_M = \begin{bmatrix}\mathbf{b}_{\frac{M}{2}}^T & \mathbf{b}_{\frac{M}{2}}^T + \left(\frac{M}{2}\right)\mathbf{1}_{\frac{M}{2}}^T\end{bmatrix}^T$ and $\mathbf{b}_{\frac{M}{2}} + \mathbf{J}\mathbf{b}_{\frac{M}{2}} = \left(\frac{M}{2} - 1\right)\mathbf{1}_{\frac{M}{2}}$, we have

$$\mathbf{E}_0\mathbf{b}_M = \frac{1}{\sqrt{2}}\left[\begin{array}{c}\hat{\mathbf{U}}_0\left(\mathbf{b}_{\frac{M}{2}} + \mathbf{J}\mathbf{b}_{\frac{M}{2}} + \frac{M}{2}\mathbf{1}_{\frac{M}{2}}\right)\\ \hline \hat{\mathbf{V}}_0\left(\mathbf{J}\mathbf{b}_{\frac{M}{2}} - \mathbf{b}_{\frac{M}{2}} - \frac{M}{2}\mathbf{1}_{\frac{M}{2}}\right)\end{array}\right] = \left[\begin{array}{c}\frac{M-1}{2}c_0\mathbf{e}_0\\ \hline \times\end{array}\right]$$

which indicates that the first $\frac{M}{2}$ equations in (4.1) are automatically satisfied, and (4.1) reduces to

$$\sum_{m=1}^{L}\hat{\mathbf{V}}_m\mathbf{e}_0 = \frac{\sqrt{2}}{c_0 M}\begin{bmatrix}-\hat{\mathbf{V}}_0\mathbf{J} & \hat{\mathbf{V}}_0\end{bmatrix}\mathbf{b}_M, \qquad (4.2)$$

which is a condition on the 0th columns of the $\hat{\mathbf{V}}_m$. In essence, we have obtained an alternative characterization of structurally regular synthesis bank using dyadic-based structures, with an equivalent but simpler condition (4.2) to impose (c.f. [7, Cond. A_{02}]). Due to limited space, we skip analysis bank of higher regularity, but the result can be similarly derived.

5. CONCLUSION

Using a dyadic-based structure which is minimal, we have established the framework for *structurally* regular BOFBs with length constraint, and identified the connection between the dyadic-based structure and the lattice structure commonly used for the design and implementation of GLBT. A reduced-parameter representation of the GLBT follows naturally. Regularity conditions on the dyadic-based structure are presented and specialized so as to accommodate linear phase. Design examples are given.

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