A ROBUST ITERATIVE ALGORITHM FOR RECONSTRUCTION FROM REDUNDANT FILTER BANKS

R. Bernardini, R. Rinaldo

Università di Udine
Dipartimento di Ingegneria Elettrica, Gestionale e Meccanica,
Via delle Scienze 208, Udine, Italy.

ABSTRACT

Because of their interactive nature, multimedia streams must be sent over UDP and suitable countermeasures for minimizing the effect of data loss have to be taken. Among the various proposed techniques, coding with redundant filter banks has been proposed as a mean to add robustness to the data stream. In order to avoid the computationally expensive matrix inversion necessary for signal reconstruction in presence of packet loss, an iterative reconstruction algorithm can be used. Unfortunately, the classical algorithm for iterative reconstruction does not necessarily converge if too many coefficients are lost. In this paper we propose a modified version which converges even in the case of excessive losses.

1. INTRODUCTION

Because of their interactive nature, multimedia streams must be sent over UDP and suitable countermeasures for minimizing the effect of data loss have to be taken. Most of the proposed approaches make the stream more robust by adding some redundancy to it.

An effective way to achieve such a goal in coding applications is to use a redundant filter bank. The problem of reconstructing the original signal from the filter bank output can be easily solved by recognizing that the action of an oversampled filter bank can be interpreted in terms of frames [1]. More precisely, general theorems of frame theory claim that the original signal can be obtained by linearly combining a set of suitable signals (known as the *dual frame*) using as coefficients the outputs of the analysis filter bank. It can be shown that the dual frame corresponding to an oversampled filter bank has a filter bank structure [1].

If some coefficient is lost, one can reconstruct the signal by using the dual of the "subframe" obtained by deleting from the original frame the functions corresponding to the lost samples. However, the construction of the new dual frame requires a computationally expensive matrix inversion. Because of this, an iterative algorithm for frame reconstruction has been proposed in [2]. Unfortunately, the algorithm of [2], converges only if the received coefficients still correspond to analysis with a complete set of functions, but such an hypothesis is not necessarily satisfied in the case of random losses.

In this paper we propose a modified version of the algorithm of [2] which converges even if the subframe is not a frame anymore. It is shown that the modified version converges with the same velocity of the original one.

2. OVERSAMPLED FILTER BANKS AND FRAMES

In the following we will consider the notation for 1D signals, but the results are also valid for multidimensional signals and filterbanks. A possible way to achieve some resilience against packet losses is to code signal x by means of a redundant filter bank

$$y_c[n] = \sum_{m\in\mathbb{Z}} x(m) h_c[Mn - m] \tag{1}$$

where M is the sampling factor, h_c, $c = 1,\dots,N$, is the impulse response of the c-th channel and $N > M$ is the number of channels. One possible way to obtain a redundant filter bank is by oversampling, but other designs are certainly possible [1]. Eq. (1) can be interpreted as a scalar product between the input sequence and the analysis function $\phi_k = h_c^*[Mn - \cdot]$, with $k = c + nN$. In operator form, we can write $y_k = (Fx)_k \triangleq \langle x, \phi_k\rangle$. In the case of a redundant filter bank, functions ϕ_k constitute a frame, i.e., the set $\Phi \triangleq \{\phi_k\}_{k\in\mathbb{Z}}$ satisfies

$$A\|x\|^2 \le \sum_{k\in\mathbb{Z}} |\langle x, \phi_k\rangle|^2 \le B\|x\|^2, \tag{2}$$

for some constants $A, B > 0$ called the frame bounds. In particular, the first inequality in (2) guaranties stable reconstruction of the input from y_k. The problem of the reconstruction of x from y_k is the infinite-dimensional counterpart of an overdetermined linear system $\mathbf{y} = \mathbf{Fx}$ where $\mathbf{F}$ is a full rank $N \times M$ matrix with $N > M$. In the filter bank context, the counterparts of $\mathbf{y}$, $\mathbf{x}$ and $\mathbf{F}$ are, respectively, the sequence

0-7803-8484-9/04/$20.00 ©2004 IEEE

of received coefficients $\hat{y}_c[n]$, the input signal x and the linear map F associated with the analysis filter bank. As a matter of fact, for finite length inputs and FIR filters, the case we will consider in the following, one can express the synthesis filter bank operation as a finite dimension matrix-vector product [3]. It is well-known that the solution $\hat{\mathbf{x}}$ of an overdetermined system of equations can be obtained as $\hat{\mathbf{x}} = \mathbf{F}^{\dagger}\mathbf{y}$, where $\mathbf{F}^{\dagger} = (\mathbf{F}^t\mathbf{F})^{-1}\mathbf{F}^t$ is the *pseudo-inverse* of $\mathbf{F}$ [4]. Such a solution has the property to be the minimum norm input best describing $\mathbf{y}$, i.e., $||\mathbf{F}\mathbf{x} - \mathbf{y}||$ is minimum even if $\mathbf{y}$ does not belong to the space $\mathrm{Im}(\mathbf{F})$ generated by the rows of matrix $\mathbf{F}$, because, for instance, of quantization of the coefficients. In operator form, the general reconstruction formula uses the pseudo-inverse $F^{\dagger}$ of F, namely

$$\hat{x} = F^{\dagger}\hat{y} = (F^*F)^{-1}F^*\hat{y} = (F^*F)^{-1}\sum_{k\in\mathbb{Z}}\phi_k\hat{y}_k = \sum_{k\in\mathbb{Z}}\tilde{\phi}_k\hat{y}_k. \tag{3}$$

The reconstructed signal is obtained by linearly combining, with coefficients $\hat{y}_k$, functions $\tilde{\phi}_k = (F^*F)^{-1}\phi_k$, which are the infinite-dimensional counterpart of the columns of $\mathbf{F}^{\dagger}$. Set $\widetilde{\Phi} \triangleq \{\tilde{\phi}_k\}_{k\in\mathbb{Z}}$ is called the *dual frame* of $\Phi \triangleq \{\phi_k\}_{k\in\mathbb{Z}}$.

3. PROBLEM STATEMENT AND SOLUTION

In this section, in view of the fact that we are concerned with the derivation of an iterative algorithm for the reconstruction of the input given the received coefficients, we will implicitly assume that the input has finite length and that the analysis filters are FIR, so that the operator F is a finite dimension matrix. In [2], a recursive algorithm for the computation of the pseudo-inverse solution (3) is presented. Starting from a frame $\Phi \triangleq \{\phi_k\}_{k\in\mathbb{Z}}$ with bounds A, B, one can write

$$x = \frac{2}{A+B}F^*Fx + \left(\mathrm{I}_d - \frac{2}{A+B}F^*F\right)x = \frac{2}{A+B}F^*Fx + Rx.$$

It is possible to show that $||R|| < 1$ [2], so that we have

$$x = (\mathrm{I}_d - R)^{-1}\frac{2}{A+B}F^*Fx = \frac{2}{A+B}\sum_k R^k F^*Fx.$$

In general, for a generic $\hat{y}$ resulting for instance from quantization of the received coefficients $y = Fx$, it is immediate to verify that [2]

$$\hat{x} = \frac{2}{A+B}(\mathrm{I}_d - R)^{-1}F^*\hat{y} = \lim_{N\to+\infty}\frac{2}{A+B}\sum_{k=0}^{N}R^kF^*\hat{y} \tag{4}$$

is the pseudo-inverse solution (3). By simple manipulations of (4), one can express

$$\hat{x}_N = \frac{2}{A+B}\sum_{k=0}^{N}R^kF^*\hat{y}$$

as a function of $\hat{x}_{N-1}$ and derive an iterative algorithm.

In Multiple Description coding, some of the coefficients in $\hat{y}$ can be lost during transmission. Denoting by $\hat{y}_I$ the set of received coefficients, we can pretend that they are derived from the analysis of the input x with the operator F_I, obtained by deleting the rows of matrix F with indices in the complement of set I. It may happen, however, that the rows of the resulting matrix F_I are not a frame anymore, i.e., that the rows span a proper subspace of the input space. In this case, the pseudo-inverse solution requires to find a minimum norm input vector $\hat{x}$, belonging to the space generated by the rows of F_I, such that $||F_Ix - \hat{y}_I||$ is minimum. It is easy to show, by direct calculations, that this solution can be obtained as

$$\hat{x} = \frac{2}{A+B}(\mathrm{I}_d - R)^{\dagger}F_I^*\hat{y}, \quad R = \left(\mathrm{I}_d - \frac{2}{A+B}F_I^*F_I\right). \tag{5}$$

Note that, in the above expression, constants A and B can be those of the *original* frame, before row cancellation in F. It is not difficult to show that $||R|| < 1$ if the rows of F_I still constitute a frame, possibly with bounds different from A and B, while $||R|| \leq 1$ in general. In particular, if the rows of F_I span a proper subset S_I of the input space, we have $Rx = x$ for any input belonging to the space orthogonal to S_I, and $||R|| = 1$. Unfortunately, $(\mathrm{I}_d - R)$ is not invertible in this case and we cannot use the expansion (4). In the following, we present the main contribution of the paper, i.e., an iterative algorithm for the computation of $(\mathrm{I}_d - R)^{\dagger}$ in (5), when $||R|| \leq 1$.

4. AN ITERATIVE ALGORITHM FOR $(\mathbf{I} - \mathbf{R})^{\dagger}$

In this section we show an algorithm for computing $(I_d - R)^{\dagger}$, where R is symmetric and positive definite, which converges even if R has unitary eigenvalues. The first step will be to find a suitable decomposition of R which separates the "good" eigenvalues (whose absolute value is less than one), from the "bad" ones.

Lemma 1. *If R is symmetric, positive definite and $||R|| \leq 1$, then there exist (and are unique) two matrices G and H such that*

$$R = G + H \tag{6a}$$
$$G^n = G, \qquad n > 0 \tag{6b}$$
$$\lim_{n\to\infty} H^n = 0 \tag{6c}$$
$$GH = HG = 0 \tag{6d}$$

Before proving Lemma 1, observe that it is easy to verify by induction that if G and H satisfy (6), then

$$R^k = G + H^k, \qquad k \geq 1. \tag{7}$$

Proof. Since R is symmetric, one can find a unitary matrix U such that

$$R = U^t D U \tag{8}$$

where D is diagonal. Define

$$(D_1)_{ii} \triangleq \begin{cases} 0 & \text{if } D_{ii} \neq 1 \\ 1 & \text{if } D_{ii} = 1 \end{cases}, \qquad (D_{<1})_{ii} \triangleq \begin{cases} D_{ii} & \text{if } D_{ii} \neq 1 \\ 0 & \text{if } D_{ii} = 1 \end{cases} \tag{9}$$

It is easy to verify that $G \triangleq U^t D_1 U$ and $H \triangleq U^t D_{<1} U$ satisfy conditions (6). In order to show that G and H are unique, observe that from (7) it follows that $\lim_{k\to\infty} R^k = G$ which implies that G and $H = R - G$ are unique. □

Property 1. *If R, G and H satisfy (6) as in Lemma 1, then*

$$(I_d - R)^\dagger = \sum_{k=0}^{\infty} H^k \tag{10}$$

Proof. By exploiting (8) one can write $I_d - R = U^t U - U^t D U = U^t (I_d - D) U$. It can be easily verified that

$$(I_d - R)^\dagger = U^t (I_d - D)^\dagger U \tag{11}$$

where $((I_d - D)^\dagger)_{ii} = 0$ if $D_{ii} = 1$ and $((I_d - D)^\dagger)_{ii} = 1/(1 - D_{ii})$ if $D_{ii} \neq 1$. It follows that $(I_d - D)^\dagger = \sum_{k=0}^{\infty} D_{<1}^k$. By exploiting such a result in (11), one obtains $(I_d - R)^\dagger = U^t \sum_{k=0}^{\infty} D_{<1}^k U = \sum_{k=0}^{\infty} H^k$ □

The final step is to transform equation (10) into an iterative algorithm for the computation of $x = (I_d - R)^\dagger y$. A possible implementation is described by equations

$$a_0 = y \qquad b_0 = y \qquad \text{start} \tag{12a}$$

$$a_N = R a_{N-1} \qquad b_N = b_{N-1} + a_N \qquad \text{iteration} \tag{12b}$$

$$x_N = b_N - N a_N \qquad \text{end} \tag{12c}$$

Claim 1. *If a_N, b_N, y and x_N are as in (12), then* $\lim_{N\to\infty} x_N = (I_d - R)^\dagger y$.

Proof. It is easy to show by induction that $a_N = R^N y$ and $b_N = \sum_{k=0}^{N} R^k y$. By exploiting (7), one obtains

$$b_N = y + \sum_{k=1}^{N} R^k y = NGy + \sum_{k=0}^{N} H^k y, \tag{13a}$$

$$a_N = R^N y = (G + H^N) y = Gy + H^N y. \tag{13b}$$

From (12c) and (13) one obtains $x_N = \sum_{k=0}^{N} H^k y - N H^N y$. Since $\|H\| < 1$, it follows that $\lim_{N\to\infty} x_N = (I_d - R)^\dagger y$. □

By usual techniques it is possible to find the following upper bound of the approximation error

$$\|x - x_N\| \leq \|H\|^N \left(N + \frac{\|H\|}{1 - \|H\|} \right) \|y\| \tag{14}$$

Equation (14) shows that the convergence is controlled by the largest eigenvalue of H and it is comparable to the convergence of the algorithm of [2].

4.1. Implementation remarks

Note that algorithm (12) is expecially suited to the case of a sparse R, since it requires only matrix-vector products which can be efficiently computed when R is sparse.

Moreover, if the frame has been obtained by means of an oversampled filter bank, operator $R = I_d - F_I{}^* F_I$ can be easily implemented by means of the original filter bank by observing that

$$F_I{}^* y = \sum_{k \in I^c} \phi_k y_k = \sum_{k \in \mathbb{Z}} \phi_k (\chi_I y)_k = F^* \chi_I y \tag{15}$$

where $(\chi_I y)_k = y_k$ if $k \notin I$ and $(\chi_I y)_k = 0$ if $k \in I$. Remembering that operator F^* corresponds to a synthesis filter bank having the (time reversed) *frame functions* as impulse responses (note that *it is not* the *dual* bank), equation (15) shows that $F_I{}^*$ can be computed by setting to zero the components of y corresponding to the lost coefficients and feeding the result into a synthesis bank. Similarly, F_I can be implemented by running the original analysis filter bank and discarding the values corresponding to the lost coefficients. Overall, operator R can be implemented as a concatenation of a synthesis and an analysis filter bank.

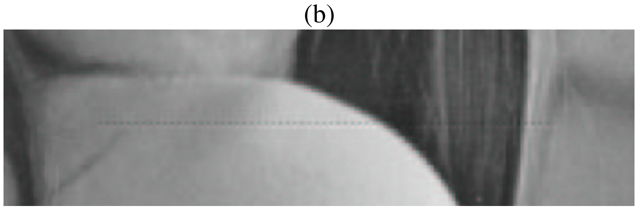
5. EXPERIMENTAL RESULTS

To illustrate the application of the theory presented above, we consider in this section a Multiple Description coding scheme for images. The filter bank has five channels, followed by factor 2 subsampling in the row and column directions. The first four filters have impulse responses $h_{i,j}(m,n) = \delta(m-i)\delta(n-j)$, $i = 0,1$, $j = 0,1$, while the fifth filter is low-pass and obtained as the separable extension of the well known Daubechies's wavelet filter with length 4 [2]. Thus, four subimages are obtained with dimension 1/4 of that of the original image and corresponding to its spatial polyphase components. An additional subimage is obtained by low-pass filtering and subsampling by a factor 2 in the row and column directions. The coding scheme has a redundancy 5/4. It is possible to show that the filterbank corresponds to a frame expansion with bounds $A = 1$, $B = 2$.

Starting from a 512 × 512 input image, each of the five 256 × 256 subimages is divided into slices of 8 × 64 pixels, which are sent as packets over the network. Each of the packets is lost independently with probability P_e. At the receiver, the iterative algorithm outlined in Section 4 is applied to the received coefficients to reconstruct the image. Fig. 1.a shows the image "Lenna" after one step of the reconstruction algorithm and a loss probability $P_e = 0.02$. The areas corresponding to lost packets are clearly visible. In the same figure, we show the reconstructed image after 300 iterations. In Fig. 1.c we show a detail of the reconstructed image after 300 iterations. The detail is positioned below the chin, where packet loss had incurred. Fig. 2 reports the

(a)

(b)



(c)

Fig. 1. (a) Recovered image "Lenna" after one step ($P_e = 0.02$) (b) Recovered image "Lenna" after 300 steps ($P_e = 0.02$), (c) A detail of the reconstructed image after 300 iterations (packet loss positioned below the chin, $P_e = 0.02$)

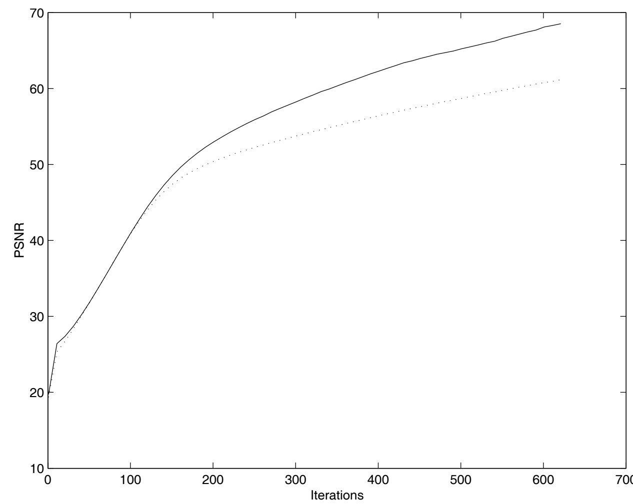


Fig. 2. PSNR vs. Number of Iterations of the iterative algorithm for $P_e = 0.01$ (solid line) and $P_e = 0.02$ (dotted line)

average PSNR, in 10 independent experiments, between the original and reconstructed images as a function of the number of iterations for two values of the probability of error, i.e., $P_e = 0.02$ and $P_e = 0.01$. It is seen that the convergence is quite slow, and, to have a reasonable complexity, it is important to apply the reconstruction only to image regions affected by losses.

6. CONCLUSIONS

A robust iterative algorithm for reconstruction from redundant filter banks has been presented. The advantage of the presented algorithm, with respect to the usual iterative algorithm for frame reconstuction, is the fact that it converges to the least square solution even in the case of excessive losses. It has been shown that the proposed algorithm converges with the same velocity of the original one.

7. REFERENCES

[1] Zoran Cvetković and Martin Vetterli, "Oversampled filter banks," *IEEE Transactions on Signal Processing*, vol. 46, no. 5, pp. 1245–1255, May 1998.

[2] Ingrid Daubechies, *Ten Lecturs on Wavelets*, SIAM, Philadelphia, 1992.

[3] M. Vetterli and J. Kovačević, *Wavelets and Subband Coding*, Signal Processing. Prentice-Hall, Englewood Cliffs, NJ, 1995.

[4] G. Strang, *Linear Algebra and its Applications*, Academic Press, New York, 1980.