ON THE FACTORIZATION OF TWO-DIMENSIONAL PARAUNITARY FILTER BANKS

Farshid Delgosha and Faramarz Fekri

Center for Signal & Image Processing
Georgia Institute of Technology, Atlanta, GA 30332-0250
E-mail: {farshid, fekri}@ece.gatech.edu

ABSTRACT

The factorization of two-dimensional (2D) FIR paraunitary filter banks is addressed in this paper. Our work is a generalization of the factorization algorithm for one-dimensional (1D) paraunitary matrices. We will present a complete factorization for multichannel, two-dimensional, FIR paraunitary filter banks. The main idea is considering a bivariate FIR matrix as a univariate polynomial whose coefficients are matrices with univariate polynomial entries. With this representation, a generalized version of the factorization algorithm for the 1D case, developed in this paper, can be used. In this direction, a new definition for paraunitary matrices is proposed and a new degree-one building block is presented. The final result is a building block that generates all 2D FIR paraunitary matrices.

1. INTRODUCTION

Filter banks over the complex field are extensively studied because of their many applications in signal processing [1–5]. After designing the filter bank for a specific application, it must be realized. One of the realization methods is factoring the polyphase matrices of the analysis and synthesis banks into the product of a fully-parameterized building block. This method gives a minimal realization in the number of the delay elements used to represent the system, it is easy to implement in practice, it covers the whole family of paraunitary matrices, and it is less sensitive to the quantization noise of the parameters of the building block [2]. When the polyphase matrices are paraunitary, there exist a degree-one building block that generates them. However, generalizing this building block to two or higher dimensions has been an open problem. This paper addresses this problem. We will generalize the definition of paraunitary matrices, introduce a new building block, and give a complete factorization. Due to lack of any general realization technique for two-dimensional filter banks, the subclass of separable paraunitary matrices are extensively used in image processing and other applications. Algorithms provided in this paper broaden the range of choice in designing two-dimensional filter banks. More details about this work and an extension to dimensions higher than two can be found in [6].

In the rest of this section, previous work in the factorization and realization of multidimensional (MD) filter banks is briefly reviewed. In Section 2, a complete factorization for a subclass of 2D paraunitary matrices is proposed. Results of this section will be used in Section 3 where a new 2D degree-one building block and a modified factorization algorithm are introduced. An illuminative numerical example is also provided in this section. Finally, Section 4 gives the concluding remarks.

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1.1. Previous Work

To our knowledge, no complete factorization for MD paraunitary matrices has been proposed yet. However, there are simple ways, e.g., Kronecker product [7], to design the special class of separable MD paraunitary matrices. This method can be easily generalized to higher dimensions, but it is not complete. Considering a 2D matrix with polynomial entries as a matrix polynomial is discussed in [4, 8, 9]. A first-level state-space realization is presented in [9]. This approach results in a complicated algorithm and a non-minimal realization. The primitive factorization technique presented in [4] finds the greatest common divisor of all matrix coefficients and extracts it out as a factor. It is proved in [4] that every 2D matrix polynomial has a unique primitive factorization. In fact, a factorization technique is presented, but the factors do not have a parameterized structure. It is conjectured in [5] that every 2D paraunitary matrix can be expressed as a product of FIR degree-one building blocks in two variables. However, as discussed in [10], this does not give a complete factorization. It is shown in [11] that the 2D transfer scattering matrix can be factored if there exist solutions to a set of linear simultaneous equations. However, this does not give a general approach. An ad hoc technique is proposed in [12] to design 2D filter banks with triangular support, but it is not a general method to factorize or generate all paraunitary matrices.

1.2. Notation

1. *Notation:* The symbol $\mathbf{A}^\dagger$ represents the conjugate transpose of the matrix $\mathbf{A}$. The expression “matrix $\mathbf{A}$ over the set $\mathcal{S}$” means that matrix $\mathbf{A}$ takes all its entries from $\mathcal{S}$.
2. *Ring of Polynomials:* The ring of univariate polynomials with complex coefficients and monomials consisting of non-negative powers of x is denoted by $\mathbb{C}[x^+]$. Rings $\mathbb{C}[x^-]$ and $\mathbb{C}[x^\pm]$ are defined similarly. Similar definitions exist for the ring of bivariate polynomials in x and y.
3. *Tilde Operator:* If $\mathbf{E}(x)$ is a 1D matrix, then $\tilde{\mathbf{E}}(x) \triangleq \mathbf{E}^\dagger(1/x^*)$. Similarly, $\tilde{\mathbf{E}}(x,y) \triangleq \mathbf{E}_*^T(x^{-1},y^{-1})$.
4. *Paraunitary in a Ring:* This is a natural generalization of the paraunitary (PU) property. Suppose $\mathbf{E}(x,y)$ is a matrix over $\mathbb{C}[x^\pm, y^\pm]$. It is called PU in $\mathbb{C}[x^\pm]$ if

$$\forall x,y, \quad \tilde{\mathbf{E}}(x,y)\mathbf{E}(x,y) = \tilde{\alpha}(x)\alpha(x)\mathbf{I}, \tag{1}$$

where $\alpha(x) \in \mathbb{C}[x^\pm]\backslash\{0\}$.

5. *1D Paraunitary FIR Degree-One Building Block:* The generating building block for FIR paraunitary matrices is [2]

$$\mathbf{D}_{\mathrm{FIR}}(t;\,\mathbf{v}) \triangleq \mathbf{I} - \mathbf{v}\mathbf{v}^{\dagger} + \mathbf{v}\mathbf{v}^{\dagger}\,t^{-1}, \tag{2}$$

where $\mathbf{v}$ is a column vector over $\mathbb{C}$ such that $\mathbf{v}^{\dagger}\mathbf{v} = 1$.

6. *1D Paraunitary IIR Degree-One Building Block:* The generating building block for IIR paraunitary matrices is [2]

$$\mathbf{D}_{\mathrm{IIR}}(t;\,\mathbf{v},a) = \mathbf{I} - \mathbf{v}\mathbf{v}^{\dagger} + \mathbf{v}\mathbf{v}^{\dagger}\left(\frac{-a^{*}+t^{-1}}{1-at^{-1}}\right), \tag{3}$$

where $\mathbf{v}$ is a column vector over $\mathbb{C}$ such that $\mathbf{v}^{\dagger}\mathbf{v} = 1$ and $a \in \mathbb{C}$.

2. FACTORIZATION OF A SUBCLASS OF 2D PARAUNITARY FILTER BANKS

Let $\mathbf{E}(x,y)$ be a 2D transfer matrix that is PU in $\mathbb{C}$ and FIR with respect to y, but in general, IIR in x as follows

$$\mathbf{E}(x,y) = \frac{1}{p(x)}\mathbf{E}_0(x) + \frac{1}{p(x)}\mathbf{E}_1(x)\,y^{-1}, \tag{4}$$

where $p(x) \in \mathbb{C}[x^{-}]\backslash\{0\}$. The following lemma, proved in [13], gives a complete factorization when $p(x)$ is a nonzero constant.

Lemma 1. *Consider $\mathbf{E}(x,y)$ in (4). Suppose $\mathbf{E}_0(x)$ and $\mathbf{E}_1(x)$ are matrices over $\mathbb{C}[x^{-}]$ that do not have any common factor of the form x^{i}. Let $p(x)$ be a nonzero constant, and N_x and N_y be degrees of $\mathbf{E}(x,y)$ with respect to x and y, respectively. $\mathbf{E}(x,y)$ is PU in $\mathbb{C}$ if and only if it can be factorized as*

$$\begin{aligned}\mathbf{E}(x,y) = \left[\prod_{i=1}^{J}\mathbf{D}_{\mathrm{FIR}}(x;\,\mathbf{v}_i)\right]\mathbf{A}_l\left[\prod_{i=1}^{N_y}\mathbf{D}_{\mathrm{FIR}}(y;\,\mathbf{u}_i)\right]\\ .\mathbf{A}_r\left[\prod_{i=J+1}^{N_x}\mathbf{D}_{\mathrm{FIR}}(x;\,\mathbf{v}_i)\right],\end{aligned} \tag{5}$$

where $\mathbf{A}_l$ and $\mathbf{A}_r$ are either identity or unitary matrices over $\mathbb{C}$ and $1 \le J \le N_x$.

When $p(x)$ is not a constant, the following lemma is applied [6].

Lemma 2. *Consider $\mathbf{E}(x,y)$ in (4) with all the conditions given in Lemma 1. Assume $p(x) \in \mathbb{C}[x^{-}]\backslash\{0\}$ has order L and $\mathbf{E}_0(x)$ and $\mathbf{E}_1(x)$ do not have any common factor that can be cancelled out by $p(x)$. $\mathbf{E}(x,y)$ is PU in $\mathbb{C}$ if and only if it can be factorized as*

$$\begin{aligned}\mathbf{E}(x,y) = \left[\prod_{i=1}^{J}\mathbf{D}_{\mathrm{IIR}}(x;\,\mathbf{v}_i,a_i)\right]\mathbf{F}(x,y)\\ .\left[\prod_{i=J+1}^{L}\mathbf{D}_{\mathrm{IIR}}(x;\,\mathbf{v}_i,a_i)\right],\end{aligned} \tag{6}$$

where $\mathbf{F}(x,y)$ can be represented by (5) and $1 \le J \le L$.

These two lemmas will be used in the next section to give a complete factorization for all 2D paraunitary matrices.

3. FACTORIZATION OF TWO-DIMENSIONAL FIR PARAUNITARY FILTER BANKS

In this section, a complete factorization for all $K \times K$ paraunitary matrices in $\mathbb{C}$ with entries from the ring $\mathbb{C}[x^{-}, y^{-}]$ is presented.

First, the matrix $\mathbf{E}(x,y)$ is expressed as a polynomial in y^{-1} whose coefficients are matrices over $\mathbb{C}[x^{-}]$,

$$\mathbf{E}(x,y) = \sum_{j=0}^{L_y}\mathbf{E}_j(x)\,y^{-j}, \tag{7}$$

where $\mathbf{E}_j(x) \triangleq \sum_{i=0}^{L_x}\mathbf{E}_{ij}\,x^{-i}$ for $j = 0,\ldots L_y$. By the PU property of $\mathbf{E}(x,y)$ we have

$$\left[\sum_{j=0}^{L_y}\tilde{\mathbf{E}}_j(x)\,y^{j}\right]\left[\sum_{k=0}^{L_y}\mathbf{E}_k(x)\,y^{-k}\right] = \mathbf{I}.$$

From here, we obtain

$$\tilde{\mathbf{E}}_{L_y}(x)\mathbf{E}_0(x) = \mathbf{0}. \tag{8}$$

Now, to factorize $\mathbf{E}(x,y)$ in (7) a new building block is defined that is degree-one with respect to y and PU in $\mathbb{C}[x^{\pm}]$.

3.1. 2D Degree-One PU Building Block in $\mathbb{C}[x^{\pm}]$

Consider the following 2D matrix polynomial

$$\mathbf{U}(y;\,\mathbf{v}(x)) \triangleq \alpha(x)\mathbf{I} - \mathbf{v}(x)\tilde{\mathbf{v}}(x) + \mathbf{v}(x)\tilde{\mathbf{v}}(x)\,y^{-1}, \tag{9}$$

where $\mathbf{v}(x)$ is a nonzero column vector over $\mathbb{C}[x^{\pm}]$ and $\alpha(x) \triangleq \tilde{\mathbf{v}}(x)\mathbf{v}(x) \in \mathbb{C}[x^{\pm}]\backslash\{0\}$. A direct computation shows that the introduced matrix is PU in $\mathbb{C}[x^{\pm}]$, i.e.,

$$\tilde{\mathbf{U}}(y;\,\mathbf{v}(x))\mathbf{U}(y;\,\mathbf{v}(x)) = \alpha^{2}(x)\mathbf{I}. \tag{10}$$

The following lemma, proved in [6], gives the determinant of this matrix.

Lemma 3. *If $\mathbf{U}(y;\,\mathbf{v}(x))$ is the $K \times K$ paraunitary matrix defined in (9), then* $\det\mathbf{U}(y;\,\mathbf{v}(x)) = \alpha^{K}(x)y^{-1}$.

$\mathbf{U}(y;\,\mathbf{v}(x))$ can be realized by only one delay element in y as the block diagram of Fig. 1. It will be proved in the next subsection that $\mathbf{U}(y;\,\mathbf{V}(x))$ is the most general 2D degree-one building block in y that is PU in $\mathbb{C}[x^{\pm}]$.

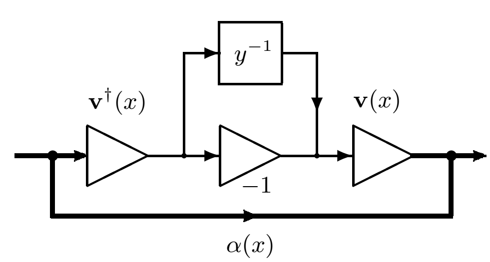

Fig. 1. 2D FIR degree-one building block in y that is PU in $\mathbb{C}[x^{\pm}]$.

The next step of factorization is the degree reduction that is explained in the next subsection.

3.2. Degree Reduction Step

The following lemma gives a complete factorization for all 2D paraunitary matrices.

Proposition 1. *Every two-dimensional FIR matrix $\mathbf{E}(x,y)$ that is paraunitary in $\mathbb{C}$ can be factorized as*

$$\mathbf{E}(x,y)=\left[\frac{1}{\alpha_{N_y}(x)}\mathbf{U}(y;\,\mathbf{v}_{N_y}(x))\right]\dots\left[\frac{1}{\alpha_1(x)}\mathbf{U}(y;\,\mathbf{v}_1(x))\right] \cdot\left[\frac{1}{\prod_{i=1}^{N_y}\alpha_i(x)}\mathbf{F}(x)\right]. \quad (11)$$

Here, N_y is the degree of $\mathbf{E}(x,y)$ with respect to y, $\mathbf{v}_i(x)$'s are nonzero vector polynomial over $\mathbb{C}[x^{\pm}]$, and $\mathbf{F}(x)$ is a matrix polynomial over $\mathbb{C}[x^{\pm}]$. In (11), all terms inside brackets are PU in $\mathbb{C}$.

Remark. This proposition gives a sufficient condition for the factorization. If $\mathbf{E}(x,y)$ is a two-dimensional FIR paraunitary matrix, then it can be factored as (11). However, given the factorization of (11), we can only conclude that $\mathbf{E}(x,y)$ is PU, but not necessarily FIR.

Proof. Similar to (7), write $\mathbf{E}(x,y)$ as a matrix polynomial in y. Let $\det\mathbf{E}(x,y)=\beta x^{-N_x}y^{-N_y}$ where N_x and N_y are the degrees of $\mathbf{E}(x,y)$ with respect to x and y, respectively [2]. Define $\mathbf{E}_{N_y}(x,y)\triangleq\mathbf{E}(x,y)$. Since $\mathbf{E}_{N_y}(x,y)$ is PU, from (8), we know there exists a nonzero vector $\mathbf{v}_{N_y}(x)$ over $\mathbb{C}[x^{\pm}]$ such that

$$\tilde{\mathbf{v}}_{N_y}(x)\mathbf{E}_0(x)=\mathbf{0}. \quad (12)$$

Define $\mathbf{E}_{N_y-1}(x,y)$ as follows

$$\mathbf{E}_{N_y-1}(x,y)\triangleq\tilde{\mathbf{U}}(y;\,\mathbf{v}_{N_y}(x))\mathbf{E}_{N_y}(x,y). \quad (13)$$

Here, the only noncausal term with respect to y is $\mathbf{v}_{N_y}(x)\tilde{\mathbf{v}}_{N_y}(x)$ $\mathbf{E}_0(x)\,y$. This term can be eliminated if $\tilde{\mathbf{v}}_{N_y}(x)$ is chosen as suggested in (12). Multiplying both sides of (13) by $\mathbf{U}(y;\,\mathbf{v}_{N_y}(x))$, we have

$$\alpha^2_{N_y}(x)\mathbf{E}_{N_y}(x,y)=\mathbf{U}(y;\,\mathbf{v}_{N_y}(x))\mathbf{E}_{N_y-1}(x,y). \quad (14)$$

Now, one can easily show

$$\tilde{\mathbf{E}}_{N_y-1}(x,y)\mathbf{E}_{N_y-1}(x,y)=\alpha^2_{N_y}(x)\mathbf{I}, \quad (15)$$

which implies that $\mathbf{E}_{N_y-1}(x,y)$ is PU in $\mathbb{C}[x^{\pm}]$. By Lemma 3, we have $\det\mathbf{E}_{N_y-1}(x,y)=\beta\alpha^{-K}_{N_y}(x)x^{-N_x}y^{-(N_y-1)}$. Therefore, the degree of $\mathbf{E}_{N_y-1}(x,y)$ with respect to y is less than that of $\mathbf{E}_{N_y}(x,y)$ by one because a degree-one building block with respect to y is successfully extracted. $\mathbf{E}_{N_y-1}(x,y)$ is PU in $\mathbb{C}[x^{\pm}]$, so it is possible to extract a degree-one building block from it. This procedure can be continued until all degree-one building blocks with respect to y are extracted. After N_y times, the remainder $\mathbf{E}_0(x,y)$ will have determinant $\det\mathbf{E}_0(x,y)=\beta x^{-N_x}$ $\prod_{i=1}^{N_y}\alpha_i^{-K}(x)$ that implies the degree of $\mathbf{E}_0(x,y)$ with respect to y is zero. Hence, $\mathbf{E}_0(x,y)$ is independent of y, so it is denoted by $\mathbf{F}(x)$. In summary, we have found the following factorization

$$\alpha^2_1(x)\dots\alpha^2_{N_y}(x)\mathbf{E}(x,y)=\mathbf{U}(y;\,\mathbf{v}_{N_y}(x))\dots\mathbf{U}(y;\,\mathbf{v}_1(x)) .\mathbf{F}(x). \quad (16)$$

After dividing both sides of (16) by $\alpha^2_1(x)\dots\alpha^2_{N_y}(x)$ and rearranging terms, we get the desired result. □

If $\alpha_i(x)=c_i\in\mathbb{C}\backslash\{0\}$ for some i, then using Lemma 1, we can write

$$\frac{1}{c_i}\mathbf{U}(y;\,\mathbf{v}_i(x))=x^{m_i}\left[\prod_{j=1}^{J_i}\mathbf{D}_{\mathrm{FIR}}(x;\,\mathbf{u}_{i,j})\right]\mathbf{A}_{l,i} .\mathbf{D}_{\mathrm{FIR}}(y;\,\mathbf{w}_i)\mathbf{A}_{r,i}\left[\prod_{j=J_i+1}^{N_i}\mathbf{D}_{\mathrm{FIR}}(x;\,\mathbf{u}_{i,j})\right], \quad (17)$$

where m_i is a nonnegative constant. When $\alpha_i(x)$ is a polynomial, we can use Lemma 2 to obtain the factorization

$$\frac{1}{\alpha_i(x)}\mathbf{U}(y;\,\mathbf{v}_i(x))=\left[\prod_{j=1}^{J_i}\mathbf{D}_{\mathrm{IIR}}(x;\,\mathbf{u}_{i,j},a_{i,j})\right]\mathbf{W}_i(x,y) \cdot\left[\prod_{j=J_i+1}^{N_i}\mathbf{D}_{\mathrm{IIR}}(x;\,\mathbf{u}_{i,j},a_{i,j})\right], \quad (18)$$

where $\mathbf{W}_i(x,y)$ is an FIR paraunitary matrix that has order one with respect to y, so it can be further factorized using Lemma 1.

If $\alpha_i(x)=c_i\in\mathbb{C}$ for all $i=1,\dots,N_y$, then $\mathbf{F}(x)/\prod_{i=1}^{N_y}c_i$ is FIR and PU in $\mathbb{C}$. Therefore, it accepts one-dimensional FIR factorization [2]. Otherwise, if at least one of $\alpha_i(x)$ is a polynomial, then $\mathbf{F}(x)/\prod_{i=1}^{N_y}\alpha_i(x)$ will be IIR, so the IIR factorization technique of [2] can be used. Therefore, considering Proposition 1, the following theorem can be proved [6].

Theorem 1. *Every two-dimensional FIR matrix $\mathbf{E}(x,y)$ that is PU in $\mathbb{C}$ can be factorized as*

$$\mathbf{E}(x,y)=x^m\prod_{i=1}^{N}\mathbf{A}_i\,\mathbf{D}_{\mathrm{IIR}}(x;\,\mathbf{v}_i,a_i)\,\mathbf{D}_{\mathrm{IIR}}(y;\,\mathbf{u}_i,b_i), \quad (19)$$

where m and N are integers, and $\mathbf{A}_i$'s are either identity or unitary matrices. Here, $\mathbf{v}_i$'s and $\mathbf{u}_i$'s are either zero or unit-norm vectors and $a_i,b_i\in\mathbb{C}$.

The following fact is the immediate result of this theorem.

Fact 1. *The general building block for two-dimensional FIR paraunitary matrices is $\mathbf{A}\,\mathbf{D}_{\mathrm{IIR}}(x;\,\mathbf{v},a)\,\mathbf{D}_{\mathrm{IIR}}(y;\,\mathbf{u},b)$.*

There are some remarks about Theorem 1 that are worth mentioning.

1. The appearance of IIR building blocks in the factorization of FIR paraunitary matrices may seem unusual. However, it is very similar to the factorization of polynomials over the ring of integers, which is not always possible. It becomes possible when we consider the polynomial over the complex field that contains the ring of integers as a subalgebra, but the factors will have complex coefficients. By multiplying some adjacent factors, we might get a new factorization into the product of irreducible polynomials over the ring of integers. In the case of the factorization of Theorem 1, we may get FIR factors if some adjacent IIR factors are multiplied.
2. Practically, there are interests in the problem of generating FIR paraunitary systems by cascading the elementary building blocks. It would be ideal to find necessary and sufficient conditions to ensure that the product of IIR factors in Theorem 1 are FIR systems. The following lemma gives a sufficient condition [6].

Fact 2. *The expression* $\mathbf{D}_{\mathrm{IIR}}(x;\,\mathbf{u},a)\mathbf{D}_{\mathrm{IIR}}(x;\,\mathbf{v},1/a^*)$ *is FIR paraunitary if and only if* $\mathbf{u}\mathbf{u}^\dagger\mathbf{v}\mathbf{v}^\dagger=\mathbf{u}\mathbf{u}^\dagger=\mathbf{v}\mathbf{v}^\dagger$.

3. The factorization of Theorem 1 is minimal with respect to y, but not necessarily with respect to x.
4. If a is a root of the polynomial $\alpha(x)$ in (9), then $1/a^*$ is another root. It means that some IIR factors in the factorization of Theorem 1 are unstable. However, as stated before, it is possible to multiply out some adjacent IIR factors and get FIR terms.

The following example shows how to utilize the proposed factorization technique.

Example 1. *Consider the 2D paraunitary matrix*

$$\mathbf{E}(x,y)=\frac{1}{2}\begin{bmatrix}x^{-1}y^{-1}(1-x^{-1}) & 1+x^{-1}\\ x^{-2}y^{-2}(1+x^{-1}) & x^{-1}y^{-1}(1-x^{-1})\end{bmatrix}.$$

It is easily verified that $\det\mathbf{E}(x,y)=-x^{-3}y^{-2}$. *Thus, it has degrees three and two with respect to* x *and* y, *respectively. Writing* $\mathbf{E}(x,y)$ *as a matrix polynomial in* y, *we get*

$$\mathbf{E}(x,y)=\underbrace{\frac{1}{2}\begin{bmatrix}0 & 1+x^{-1}\\ 0 & 0\end{bmatrix}}_{\mathbf{E}_0(x)}+\frac{1}{2}\begin{bmatrix}x^{-1}-x^{-2} & 0\\ 0 & x^{-1}-x^{-2}\end{bmatrix}$$
$$\cdot y^{-1}+\frac{1}{2}\begin{bmatrix}0 & 0\\ x^{-2}+x^{-3} & 0\end{bmatrix}y^{-2}.$$

The unit-norm vector $\mathbf{v}_2(x)=[0\quad 1]^T$ *is orthogonal to* $\mathbf{E}_0(x)$ *from the left. Therefore, the factor* $\mathbf{U}(y;\,\mathbf{v}_2(x))$ *can be extracted from the left of* $\mathbf{E}(x,y)$, *so*

$$\mathbf{E}(x,y)=\begin{bmatrix}1 & 0\\ 0 & y^{-1}\end{bmatrix}\underbrace{\frac{1}{2}\begin{bmatrix}x^{-1}y^{-1}(1-x^{-1}) & 1+x^{-1}\\ x^{-2}y^{-1}(1+x^{-1}) & x^{-1}(1-x^{-1})\end{bmatrix}}_{\mathbf{E}_1(x,y)}.$$

Writing $\mathbf{E}_1(x,y)$ *as a matrix polynomial, we will see that the factor* $\mathbf{U}(y;\,[1\quad 0]^T)$ *can be extracted from the right, so*

$$\mathbf{E}_1(x,y)=\underbrace{\frac{1}{2}\begin{bmatrix}x^{-1}(1-x^{-1}) & 1+x^{-1}\\ x^{-2}(1+x^{-1}) & x^{-1}(1-x^{-1})\end{bmatrix}}_{\mathbf{F}(x)}\begin{bmatrix}y^{-1} & 0\\ 0 & 1\end{bmatrix}.$$

Here, $\mathbf{F}(x)$ *is a one-dimensional paraunitary matrix in* $\mathbb{C}$. *Hence, it can be factorized. Writing the factors obtained in each step next to each other, we get the factorization*

$$\mathbf{E}(x,y)=\begin{bmatrix}1 & 0\\ 0 & y^{-1}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & x^{-1}\end{bmatrix}\begin{bmatrix}\frac{1-x^{-1}}{2} & \frac{1+x^{-1}}{2}\\ \frac{1+x^{-1}}{2} & \frac{1-x^{-1}}{2}\end{bmatrix}$$
$$\cdot\begin{bmatrix}x^{-1} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}y^{-1} & 0\\ 0 & 1\end{bmatrix}.$$

4. CONCLUSION

A complete factorization for the class of two-dimensional FIR paraunitary matrices was presented in this paper. The approach considers a 2D paraunitary matrix as a 1D one whose coefficients are polynomial matrices, and then uses a generalized version of the 1D factorization algorithm. In this direction, the definition of paraunitariness is generalized and a new building block is defined that has degree one in one of the variables and PU in the ring of polynomials of the other variable. The generalized factorization algorithm gives a first-level factorization in only one of the variables. The next step is to factor each term obtained in the first step. Since each factor has degree one with respect to one of the variables, the problem reduces to factoring a special family of two-dimensional paraunitary matrices. There already exist a factorization for a subclass of this family. We completed this factorization and gave a complete factorization for the whole family. After putting all these results together, we gave a fully-parameterized building block for the whole family of two-dimensional FIR paraunitary matrices.

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