# LINEAR PHASE OVERSAMPLED FILTER BANKS

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## ABSTRACT

This paper introduces a factorization for the design of oversampled filter banks with linear phase and complex-valued coefficients. Our approach is based on the design of rectangular paraunitary polyphase matrices and gives a general formulation for perfect reconstruction filter banks with uniform sampling in the subbands, critically sampled or oversampled, and linear phase filters with real- or complex-valued coefficients. By using a framework that is not restricted to modulation of a prototype window, as in oversampled modulated filter banks, we are able to increase the design freedom and to allow the design of non uniform bandwidth oversampled filter banks.

## I. INTRODUCTION

Oversampled filter banks find application in redundant wavelet systems, signal filtering, and communication systems [1]–[4]. Our interest in oversampled filter banks is due to their additional design flexibility [5], [6] and their application in wavelet systems. Wavelet systems implemented with filter banks are useful in image processing. In such systems, the use of linear phase filters is important to reduce the artifacts around the objects in the image.

Although the theory of perfect reconstruction oversampled filter banks has been extensively explored [5], [7], [8], the design of oversampled filter banks has been restricted almost exclusively to the class of modulated filter banks. In contrast to the earlier oversampled works, in [9] we give under the same framework the design of filter banks, oversampled or critically sampled, that are not constrained by modulation of a prototype window. By removing the constraint of modulation we are able to explore the increased design freedom and to design oversampled filter banks that are not restricted, for example, to integer oversampling ratios [10] or frequency responses with uniform bandwidth. The framework in [9], based on rectangular paraunitary polyphase matrices, reveals a general approach for the design of linear phase perfect reconstruction filter banks, either oversampled or critically sampled. However, the factorization of linear phase filter banks presented in [9] is restricted to filters with real-valued coefficients.

This paper extends the results in [9] to include the factorization of oversampled filter banks with linear phase and complex-valued coefficients. The factorization of rectangular paraunitary matrices introduced in this paper permits analysis and synthesis with FIR filters and a simple formulation of the constraints for linear phase. Our factorization is derived from the factorizations proposed in [9], [11] and [5], itself an extension of [12] used in the design of critically sampled filter banks. In the critically sampled case, our factorization resembles the factorization derived in [13].

## II. RECTANGULAR PARAUNITARY MATRICES

An $M$-channel filter bank with uniform sampling factor $K$ in the subbands is shown in Fig. 1. Perfect reconstruction means that in the absence of processing, the output signal $\hat{x}[n]$ is a replica of the input signal $x[n]$. In practice, perfect reconstruction is accomplished with some delay introduced by the filters. The term oversampled refers to filter banks where the sampling factor is less than the number of channels; that is, $1 \le K < M$; the term critically sampled refers to the case $K = M$. Our formulation allows the amount of oversampling to be specified as a value in the interval from the critically sampled, in which $K = M$, to the maximum oversampled case, in which $K = 1$, meaning that no downsamplers and upsamplers are used.

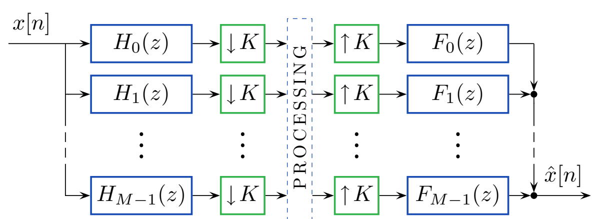


**Fig. 1**. $M$-channel filter bank with uniform sampling factor $K$.

We consider an $M$-channel perfect reconstruction filter bank with sampling factor $K$ whose filters are causal and FIR. The analysis filter bank is represented by the $M \times K$ polyphase matrix $\mathbf{E}(z)$, which represents the set of $M$ analysis filters decomposed into $K$ polyphase components each [14]. Similarly, the synthesis filter bank is represented by the matrix $z^{-P}\,\tilde{\mathbf{E}}(z)$, where $P$ is an integer and $\tilde{\mathbf{E}}(z) = \mathbf{E}^{\dagger}(1/z^{*})$. The symbol $\dagger$ represents the transpose conjugate operation, and the symbol $*$ represents the conjugate operation.

Our approach to design perfect reconstruction filter banks is to choose the matrix $\mathbf{E}(z)$ to be paraunitary [14]; that is, $\tilde{\mathbf{E}}(z)\,\mathbf{E}(z) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. If the order of the polynomials in $\tilde{\mathbf{E}}(z)$ is $P$, it is clear that $z^{-P}\,\tilde{\mathbf{E}}(z)$ will have no positive powers of $z$. In this case, both analysis and synthesis banks have causal filters, and perfect reconstruction holds with a delay $K(P+1)-1$, which implies $\hat{x}[n] = x[n-K(P+1)+1]$.

This paper introduces an approach for the design of oversampled filter banks with complex-valued coefficients and linear phase filters based on the factorization of rectangular paraunitary matrices proposed in [5]. The factorization in [5] can be used in

[1] Supported by Rice University and CNPq−Brazil.

the design of rectangular paraunitary matrices; however, it is not in a form convenient for representing linear phase constraints. This paper uses an equivalent factorization that gives a simple formulation of the constraints for linear phase with complex-valued coefficients.

The factorization in [5] can be written as [9]

$$\mathbf{E}(z) = \mathbf{U}_L\,\mathbf{\Lambda}_{m_L}(z)\,\mathbf{U}_{L-1}\,\mathbf{\Lambda}_{m_{L-1}}(z)\ldots \times \mathbf{U}_1\,\mathbf{\Lambda}_{m_1}(z)\,\mathbf{U}_0, \tag{1}$$

where $\mathbf{U}_i$, for $i = 1, 2, \ldots, L$, are unitary matrices of size $M \times M$; $\mathbf{U}_0$ is an $M \times K$ orthonormal matrix; and $\mathbf{\Lambda}_{m_i}(z)$ is the block diagonal matrix, defined by

$$\mathbf{\Lambda}_{m_i}(z) = \begin{bmatrix} \mathbf{I}_{M-m_i} & \mathbf{0} \\ \mathbf{0} & z^{-1}\,\mathbf{I}_{m_i} \end{bmatrix}, \tag{2}$$

where $\mathbf{I}_{m_i}$ is an identity matrix of size $m_i \times m_i$. In the particular case $K = M$, the factorization in (1) corresponds to the factorization given in [13].

To provide an easy formulation of the constraints for the design of filter banks with linear phase filters, (1) is written in the equivalent form [9], [15]

$$\mathbf{E}(z) = \mathbf{Q}_L\mathbf{\Lambda}_{m_L}(z)\mathbf{Q}_L^\dagger\mathbf{Q}_{L-1}\mathbf{\Lambda}_{m_{L-1}}(z)\mathbf{Q}_{L-1}^\dagger \ldots \times \mathbf{Q}_1\mathbf{\Lambda}_{m_1}(z)\mathbf{Q}_1^\dagger\mathbf{Q}_0, \tag{3}$$

where $\mathbf{\Lambda}_{m_i}(z)$, for $i=1, 2, \ldots, L$, is defined in (2), and $\mathbf{Q}_i = \mathbf{U}_L\,\mathbf{U}_{L-1}\ldots\mathbf{U}_i$, for $i=0, 1, \ldots, L$, such that $\mathbf{Q}_i$ is unitary, of size $M \times M$, for $i > 0$, and of size $M \times K$, for $i = 0$. The square factors $\mathbf{Q}_i\,\mathbf{\Lambda}(z)\,\mathbf{Q}_i^\dagger$, for $i = 1, 2, \ldots, L$, have the form

$$\mathbf{Q}_i\,\mathbf{\Lambda}_{m_i}(z)\,\mathbf{Q}_i^\dagger = \begin{bmatrix} \mathbf{U}_{i,0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{i,1} \end{bmatrix} \begin{bmatrix} \mathbf{C}_i & \mathbf{0} & \mathbf{S}_i \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{S}_i & \mathbf{0} & -\mathbf{C}_i \end{bmatrix} \times \mathbf{\Lambda}_{m_i}(z) \begin{bmatrix} \mathbf{C}_i & \mathbf{0} & \mathbf{S}_i \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{S}_i & \mathbf{0} & -\mathbf{C}_i \end{bmatrix} \begin{bmatrix} \mathbf{U}_{i,0}^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{i,1}^\dagger \end{bmatrix}, \tag{4}$$

where $\mathbf{U}_{i,0}$ and $\mathbf{U}_{i,1}$ are unitary matrices of sizes $M - m_i \times M - m_i$ and $m_i \times m_i$, respectively. The blocks $\mathbf{C}_i$ and $\mathbf{S}_i$ are diagonal matrices defined, respectively, as

$$\begin{aligned} \mathbf{C}_i &= \mathrm{diag}(\cos\theta_{i,0}, \cos\theta_{i,1}, \ldots, \cos\theta_{i,P-1}) \geq \mathbf{0}, \text{ and} \\ \mathbf{S}_i &= \mathrm{diag}(\sin\theta_{i,0}, \sin\theta_{i,1}, \ldots, \sin\theta_{i,P-1}) \geq \mathbf{0}. \end{aligned} \tag{5}$$

These matrices have size $P \times P$, where $P = \min(M - m_i, m_i)$ is the smallest of $M - m_i$ and $m_i$. The square matrices in (4) are similar to the factors of the factorization of square paraunitary matrices derived for the design of critically sampled filter banks in [13]. In the next section, we impose to (3) additional constraints for linear phase filters with real-valued coefficients.

## III. LINEAR PHASE WITH REAL COEFFICIENTS

We derive the constraints on the factorization of the polyphase matrix represented in (3) in order to obtain filters having the linear phase property with real-valued coefficients. First, we need to establish a characterization of the polyphase matrix which reflects the linear phase property of the filters. We follow the characterization used in [16] in the case of linear phase, critically sampled, and paraunitary filter banks. The paraunitary matrix corresponds to a set of filters which have linear phase that satisfies the property

$$\mathbf{E}(z) = z^{-L}\,\mathbf{D}_M\,\mathbf{E}(z^{-1})\,\mathbf{J}_K, \tag{6}$$

where $L$ is the order of the paraunitary matrix $\mathbf{E}(z)$, and $\mathbf{J}_K$ is the counter diagonal matrix of size $K \times K$ [16]. The matrix $\mathbf{D}_M$ in (6) is an $M \times M$ diagonal matrix whose diagonal entries are $+1$ or $-1$, the $+1$ in those rows which correspond to symmetric filters and the $-1$ in those rows which correspond to antisymmetric filters.

It is important to observe that the characterization in (6) is not unique; instead, it represents a large class of filter banks [16]. The filters described by (6) have the same center of symmetry at $((L+1)K-1)/2$.

In the critically sampled case, when the polyphase matrix is square, it is shown in [16] that if the number of channels $M$ is even, then there are $M/2$ symmetric and $M/2$ antisymmetric filters. However, if $M$ is odd, then there are $(M+1)/2$ symmetric and $(M-1)/2$ antisymmetric filters. In the oversampled case, the number of symmetric and antisymmetric filters does not need to follow the same division of the critically sampled case. However, due to the structure of the factorization in (3), we will assume that the number of symmetric filters is equal to the number of antisymmetric filters for $M$ even, and that the number of symmetric filters exceeds the number of antisymmetric filters by one for $M$ odd, as happens in the critically sampled case.

The linear phase property with real-valued coefficients is based on $\mathbf{E}(z)$ defined according to (3) and (4) as

$$\mathbf{E}(z) = \left(\prod_{i=L}^{1} \mathbf{U}_i\,\mathbf{X}_i\,\mathbf{\Lambda}(z)\,\mathbf{X}_i^\dagger\,\mathbf{U}_i^\dagger\right)\,\mathbf{V}_0. \tag{7}$$

The property (6) and the factorization of $\mathbf{E}(z)$ in (7) can be used to show that the number of symmetric and antisymmetric linear phase filters imposes constraints on the factor $\mathbf{V}_0$. Following the steps outlined in [16], taking the trace of both sides of (6) we can show that

$$\mathrm{tr}\left(\mathbf{D}_M\,\mathbf{V}_0\,\mathbf{V}_0^\dagger\right) = \begin{cases} 0, & \text{for } K \text{ even}, \\ 1, & \text{for } K \text{ odd}, \end{cases} \tag{8}$$

where tr($\mathbf{A}$) is the trace of matrix $\mathbf{A}$. In the particular case of the critically sampled filter bank, $\mathbf{V}_0\,\mathbf{V}_0^\dagger = \mathbf{I}$, the result in (8) corresponds to the result derived in [16]. In this case, there is an equal number of symmetric and antisymmetric filters for $M$ even, and one extra symmetric filter for $M$ odd. In the oversampled case, $\mathbf{V}_0$ is an orthonormal, non square matrix, constrained by the linear phase property. Now, we derive the constraints on $\mathbf{V}_0$, as well as the other terms in the factorization of $\mathbf{E}(z)$.

Applying the linear phase property (6) to $\mathbf{E}(z)$ results

$$z^{-L}\,\mathbf{D}_M\,\mathbf{E}(z^{-1})\,\mathbf{J}_K = \left(\prod_{i=L}^{1} \mathbf{U}_i\,\mathbf{X}_i\,\mathbf{\Lambda}(z)\,\mathbf{X}_i^\dagger\,\mathbf{U}_i^\dagger\right)\,\mathbf{D}_M\,\mathbf{V}_0\,\mathbf{J}_K. \tag{9}$$

It can be shown that if the factors of $\mathbf{E}(z)$ are appropriately chosen, depending on whether $M$ is even or odd, then the diagonal matrix $\mathbf{D}_M$ can be propagated to the right of the product between parentheses, bringing back the factors of $\mathbf{E}(z)$.

Therefore, to satisfy the linear phase property, such that (9) is equal to (7), it is required additionally that

$$\mathbf{V}_0 = \mathbf{D}_M \, \mathbf{V}_0 \, \mathbf{J}_K. \tag{10}$$

We next consider the following four possible cases for even and odd values of $M$ and $K$. The factors of $\mathbf{E}(z)$ are indicated first for $M$ even.

• <u>$M$ even</u>

For $M$ even, we assume an equal number of symmetric and antisymmetric filters, that is, $\mathbf{D}_M = \text{diag}(\mathbf{I}, -\mathbf{I})$. In this case, (9) is valid if

$$\mathbf{X}_i = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}, \mathbf{\Lambda}(z) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & z^{-1}\,\mathbf{I} \end{bmatrix}, \mathbf{U}_i = \begin{bmatrix} \mathbf{U}_{i,0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{i,1} \end{bmatrix}, \tag{11}$$

for $i = 1, 2, \ldots, L$, where all the blocks in $\mathbf{U}_i$, $\mathbf{X}_i$ and $\mathbf{\Lambda}(z)$ have size $M/2 \times M/2$. The constraints on the square factors of $\mathbf{E}(z)$, represented by $\mathbf{V}(z)$ in (7), depend on $M$ and not on $K$. The constraints on $\mathbf{V}_0$ depend on whether $K$ is even or odd.

• $K$ even.

For $K$ even, imposing (8) and (10) results

$$\mathbf{V}_0 = \left[ \begin{array}{cc} \mathbf{V}_{00} & \mathbf{V}_{00}\,\mathbf{J}_{K/2} \\ \mathbf{V}_{10} & -\mathbf{V}_{10}\,\mathbf{J}_{K/2} \end{array} \right], \tag{12}$$

where all the blocks have size $M/2 \times K/2$. The rectangular paraunitary matrix (7) with factors constrained as in (11) and (12) corresponds to a set of linear phase filters, if the matrices $\mathbf{X}_i$, $\mathbf{U}_i$, for $i = 1, 2, \ldots, L$, and $\mathbf{V}_0$ have only real entries. In the particular case that the paraunitary matrix is square, that is $K = M$, it can be verified that the constraints in (11) and (12) correspond to the constraints in [16] and [17].

• $K$ odd.

For $K$ odd, imposing (8) and (10) results

$$\mathbf{V}_0 = \left[ \begin{array}{ccc} \mathbf{V}_{00} & \mathbf{v} & \mathbf{V}_{00}\,\mathbf{J}_{(K-1)/2} \\ \mathbf{V}_{10} & \mathbf{0} & -\mathbf{V}_{10}\,\mathbf{J}_{(K-1)/2} \end{array} \right], \tag{13}$$

where $\mathbf{V}_{00}$ and $\mathbf{V}_{10}$ have size $M/2 \times (K-1)/2$. Now, we can look at the cases in which $M$ is odd.

• <u>$M$ odd</u>

For $M$ odd, we assume that the number of symmetric filters exceeds the number of antisymmetric filters by one, that is, $\mathbf{D}_M = \text{diag}(\mathbf{I}, 1, -\mathbf{I})$. In this case, (9) is valid if

$$\mathbf{X}_i = \tfrac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \sqrt{2} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} \end{bmatrix}, \ \mathbf{\Lambda}(z) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & z^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & z^{-2}\,\mathbf{I} \end{bmatrix}, \tag{14}$$

and $\mathbf{U}_i = \text{diag}(\mathbf{U}_{i,0}, \mathbf{U}_{i,1})$, for $i = 1, 2, \ldots, L$. The block $\mathbf{U}_{i,0}$ has size $(M+1)/2 \times (M+1)/2$, and the block $\mathbf{U}_{i,1}$ has size $(M-1)/2 \times (M-1)/2$. Note that we are assuming that $\mathbf{\Lambda}(z)$ has the quadratic power $z^{-2}$; therefore, the order of $\mathbf{E}(z)$ is $2L$ and not $L$ such that $z^{-L}$ in (9) must be substituted by $z^{-2L}$ when $M$ is odd. The form of the diagonal matrix $\mathbf{\Lambda}(z)$ is based on the linear phase structure for 3-channel QMF banks in [18].

The constraints on $\mathbf{V}_0$ depend on whether $K$ is even or odd.

• $K$ even.

For $K$ even, imposing (8) and (10) results in the same type of symmetry indicated in (12) but $\mathbf{V}_{00}$ has size $(M+1)/2 \times K/2$ and $\mathbf{V}_{10}$ size $(M-1)/2 \times K/2$.

• $K$ odd.

For $K$ odd, imposing (8) and (10) results in the same type of symmetry indicated in (13) but $\mathbf{V}_{00}$ has size $(M+1)/2 \times (K-1)/2$ and $\mathbf{V}_{10}$ size $(M-1)/2 \times (K-1)/2$.

An advantage of our approach for the factorization of linear phase filter banks with $M$ odd over the approaches in [16] and [17] is that the formulation in (14) generates filters of equal length. In [16] and [17] one of the symmetric filters has always shorter length then the others and, as a consequence, more limited frequency selectivity.

## IV. LINEAR PHASE WITH COMPLEX COEFFICIENTS

The formulation of oversampled filter banks with linear phase and complex-valued coefficients introduced in this paper is a generalization of the factorization of square paraunitary matrices for $M$ even introduced in [11]. Furthermore, our formulation corresponds to an extension of the factorization for the design of linear phase filter banks with real-valued coefficients presented in the previous section [9]. Our new formulation allows the design of perfect reconstruction filter banks with linear phase and complex-valued coefficients, critically sampled or oversampled, and even or odd number of channels.

The paraunitary matrix with polynomials in $z^{-1}$ and complex-valued coefficients corresponds to a set of filters which have linear phase that satisfies the property

$$\mathbf{E}(z) = z^{-L}\,\mathbf{D}_M\,\mathbf{E}(1/z^*)^*\,\mathbf{J}_K, \tag{15}$$

where $\mathbf{D}_M$ and $\mathbf{J}_K$ are defined as in (6). We assume the same division between symmetric and antisymmetric filters adopted in (6). That is, for $M$ even there is equal number of conjugate-symmetric and conjugate-antisymmetric filters, and for $M$ odd there is an extra conjugate-symmetric filter.

The linear phase property is based on $\mathbf{E}(z)$ defined according to (7). Now, however, the product $\mathbf{U}_i\,\mathbf{X}_i$, for $i = 1, 2, \ldots, L$, has complex entries. The linear phase property (15) of $\mathbf{E}(z)$ implies that

$$\mathbf{E}(z) = \mathbf{D}_M \left( \prod_{i=L}^{1} \mathbf{U}_i^*\,\mathbf{X}_i^*\,\mathbf{\Lambda}(z^{-1})\,\mathbf{X}_i^T\,\mathbf{U}_i^T \right) \mathbf{V}_0^*\,\mathbf{J}_K, \tag{16}$$

where $T$ represents the transpose operation. Proceeding as in the case of paraunitary matrices with real-valued coefficients, the matrix $\mathbf{D}_M$ is propagated to the right of the product between parentheses, bringing back the factors of $\mathbf{E}(z)$ in (7). This implies that $\mathbf{V}_0$ must satisfy the property

$$\mathbf{V}_0 = \mathbf{D}_M\,\mathbf{V}_0^*\,\mathbf{J}_K. \tag{17}$$

We next consider the following four possible cases for even and odd values of $M$ and $K$. The factors of $\mathbf{E}(z)$ are indicated first for $M$ even.

• <u>$M$ even</u>

For $M$ even, we assume an equal number of conjugate-symmetric and conjugate-antisymmetric filters. That is, $\mathbf{D}_M = \text{diag}(\mathbf{I}, -\mathbf{I})$. In this case, (16) is valid if $\mathbf{X}_i$ and $\mathbf{\Lambda}(z)$ are defined as in (11), and

$$\mathbf{U}_i = \left[ \begin{array}{cc} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & j\mathbf{I} \end{array} \right] \underbrace{\left[ \begin{array}{cc} \mathbf{Q}_{i,00} & \mathbf{Q}_{i,01} \\ \mathbf{Q}_{i,10} & \mathbf{Q}_{i,11} \end{array} \right]}_{\mathbf{Q}i} \left[ \begin{array}{cc} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & j\mathbf{I} \end{array} \right], \tag{18}$$

where $j = \sqrt{-1}$ and $\mathbf{Q}_i$ is an $M \times M$ orthogonal matrix. That is, $\mathbf{Q}_i^T \mathbf{Q}_i = \mathbf{I}$. The blocks in $\mathbf{Q}_i$, as well as, $\mathbf{U}_i$, $\mathbf{X}_i$ and $\mathbf{\Lambda}(z)$ have size $M/2 \times M/2$. Comparing the unitary matrix $\mathbf{U}_i$ defined in (18) to the orthogonal matrix $\mathbf{U}_i$ defined in (11), we see that the complex entries of $\mathbf{U}_i$ in (18) correspond to the two extra blocks $j\mathbf{Q}_{i,01}$ and $j\mathbf{Q}_{i,10}$. The form of $\mathbf{U}_i$ in (18) is inspired in [11].

For $K$ even and odd, the constraints in (8) and (17) are satisfied by (12) and (13), respectively. Now, we look at the cases in which $M$ is odd, extending the formulation in [11], which assumes $M$ even and square paraunitary matrices.

• $M$ odd

For $M$ odd, we assume that the number of conjugate-symmetric filters exceeds the number of conjugate-antisymmetric filters by one. That is, $\mathbf{D}_M = \mathrm{diag}(\mathbf{I}, 1, -\mathbf{I})$. In this case, (16) is valid if $\mathbf{X}_i$ and $\mathbf{\Lambda}(z)$ are defined as in (14), and

$$\mathbf{U}_i = \begin{bmatrix} \mathbf{I}_{\frac{M+1}{2}} & \mathbf{0} \\ \mathbf{0} & j\mathbf{I}_{\frac{M-1}{2}} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{Q}_{i,00} & \mathbf{Q}_{i,01} \\ \mathbf{Q}_{i,10} & \mathbf{Q}_{i,11} \end{bmatrix}}_{\mathbf{Q}_i} \begin{bmatrix} \mathbf{I}_{\frac{M+1}{2}} & \mathbf{0} \\ \mathbf{0} & j\mathbf{I}_{\frac{M-1}{2}} \end{bmatrix},$$

where $\mathbf{Q}_i$ is an $M \times M$ orthogonal matrix with blocks $\mathbf{Q}_{i,00}$ and $\mathbf{Q}_{i,11}$ of sizes $(M+1)/2 \times (M+1)/2$ and $(M-1)/2 \times (M-1)/2$, respectively. For $K$ even, $\mathbf{V}_0$ is defined as in (12), and for $K$ odd, $\mathbf{V}_0$ is defined as in (13).

## V. CONCLUSION

We have introduced a factorization of FIR oversampled filter banks with complex-valued coefficients and linear phase filters based on the factorization of rectangular paraunitary matrices. The new factorization is an extension of our previous result for the design of oversampled filter banks with real-valued coefficients. The framework based on rectangular paraunitary polyphase matrices reveals a general approach for the design of linear phase perfect reconstruction filter banks. This framework allows the amount of oversampling to be specified as a value from the critically sampled to maximum oversampled, meaning that no downsamplers and upsamplers are used.

After completing this paper, the authors learned of the real case, but not the complex, being addressed in [19].


## VI. ACKNOLEDGEMENT

The authors would like to thank Dr. M. Embree, Dr. R. A. Tapia and Dr. D. C. Sorensen for many useful discussions.


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