A NEW COMPLEX WAVELET TRANSFORM BY USING RI-SPLINE WAVELET

Zhong Zhang, Hisanaga Fujiwara

Industrial Technology Center of Okayama prefecture
5301 Haga Okayama-shi, 701-1296, Japan
zhang@okakogi.go.jp

Hiroshi Toda, Hiroaki Kawabata

Okayama Prefectural University
111 Soja-shi, 719-1197, Japan

ABSTRACT

In this study, we propose a new complex wavelet, the RI-spline wavelet, which is constructed using spline wavelets for dual-tree DWT. In the RI-spline wavelet, the real and imaginary components become an approximate Hilbert pair to each other. Then we propose a new dual-tree algorithm which uses an interpolation method for providing a half-sample-delay between the two filters of the trees. Finally, we experimentally show that the translation invariance, which can not be obtained by the ordinary DWT, is obtained by RI-spline wavelet.

1. INTRODUCTION

The Discrete Wavelet Transform (DWT) is a powerful signal-processing tool, for which a fast algorithm based on the Multi–Resolution Analysis (MRA) algorithm is proposed by Mallat [1]. However, the DWT has a disadvantage that the transformed result is not translation invariant. This means that shifts of the input-signal generate undesirable changes in the wavelet coefficients. So the DWT can not catch features of the signals exactly.

In order to overcome this problem, Kingsbury [2] proposed a complex wavelet transform, the Dual-Tree Wavelet Transform (DTWT), which realizes approximately translation invariance. In the two trees of the DTWT, a pair of filter banks, which is a Hilbert pair, is used. Thus approximate translation invariance can be obtained. In the DTWT, in order to get uniform intervals between samples from the two trees in or below level-1, the filters in one tree should provide a half-sample-delay (at each filter's input rate) from those in the other tree. This realization of a half-sample-delay is very difficult. For this, Kingsbury first provided the delay of one tree's filter which are one sample offset from another tree's filter in level -1. However, this means that in level -1, that is, the half frequency range, the wavelet coefficients can not be used for complex analysis. In addition, the design of Kingsbury's filter banks is complicated because it requires an iterative optimization over the space of perfect-reconstruction filter-banks, although Selesnick [3] proposed a new method which may unburden this problem.

Fernandes *et al.* [4] proposed a new framework for the implementation of Complex Wavelet Transforms (CWTs). In this framework, at first the Hilbert Transform is applied to the input signal then real and imaginary pair of the signal can be obtained, for each of which the same DWT is applied. Thus we get the real and imaginary part of the CWTs results. This approach is very simple and excellent since it can use a current DWT and requires neither designing new wavelets associated with filter banks nor providing a half-sample-delay between the two trees. However, due to using the Hilbert Transform, the computational cost increases by twice the FFT than the DTWT.

In this study, in order to overcome the fore-mentioned disadvantage of Kingsbury's DTWT without increasing the computational cost, we propose a new complex wavelet, the Real–Imaginary Spline Wavelet (RI-Spline wavelet), which is constructed using spline wavelets for the dual-tree DWT. In the dual-tree DWT using the RI-Spline wavelet, we realize a half-sample-delay between the two trees using interpolation. Using the RI-Spline wavelet, complex analysis can be carried out coherently in all analysis levels. Finally, we experimentally show how translation invariance is obtained by the DTWT using the RI-spline wavelet.

2. SPLINE WAVELET AND ITS CHARACTERICS

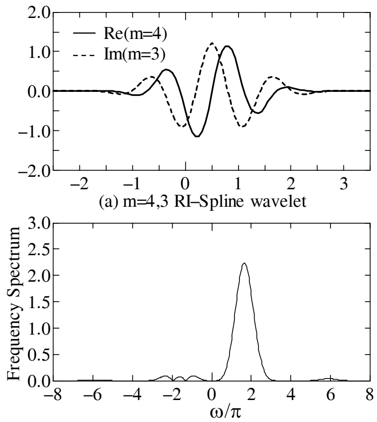
The Spline wavelet[5] is defined as follows using an integer rank m which is greater than 2.

$$\psi_m(t) = \sum_n q_n N_m(2t-n), \quad n = 0, \cdots, 3m-2, \tag{1}$$

where Spline scaling function $N_m(t)$ is computed using Eq. (2), and the coefficient q_n are computed using Eq. (3). Hereafter, we call the Spline function used for a scaling function as *Spline scaling function*.

$$N_m(t) = \frac{t}{m-1} N_{m-1}(t) + \frac{m-t}{m-1} N_{m-1}(t-1), \quad t \in \boldsymbol{R}, \tag{2}$$

$$q_n = \frac{(-1)^n}{2^{m-1}} \sum_{l=0}^{m} \binom{m}{l} N_{2m}(n+1-l), \quad n = 0, \cdots, 3m-2. \tag{3}$$



(a) m=4,3 RI–Spline wavelet

(b)Fourier transform of m=4,3 RI–Spline wavelet

Fig. 1. Example of the m=4,3 RI-Spline wavelet and its Fourier transform.

The Spline wavelet has an anti–symmetric property when m is odd number (m_o), and has a symmetric property when m is an even number (m_e). From these properties, the Spline wavelets have a generalized linear phase, and the distortion of the reconstructed signal can be minimized. The way of computing decomposition and reconstruction sequences is explained in reference [5].

Generally, Mallat's fast algorithm for DWT starts from level 0, where the signal $f(t)$ is approximated as $f_0(t)$, and the signal is decomposed by the following formula:

$$f_0(t) = \sum_k c_k^0 \phi(t-k), \quad k \in \boldsymbol{Z}, \tag{4}$$

In this equation, $\phi(t)$ means a scaling function. Usually, in the Spline wavelet, as the scaling function $N_m(t)$ is not orthogonal, the signal $f(t)$ is approximated as $f_0(t)$ using the following interpolation.

$$f_0(t) = \sum_k f(k) L_m(t-k), \quad k \in \boldsymbol{Z}, \tag{5}$$

The Fundamental Spline $L_m(t)$ of the rank m is defined as

$$L_m(t) = \sum_k \beta_k^m N_m(t + \frac{m}{2} - k), \quad k \in \boldsymbol{Z}, \tag{6}$$

which has the interpolation property $L_m(k)$=$\delta_{k,0}$, $k \in \boldsymbol{Z}$. Using Eqs. (5) and (6), we obtain the following equations.

$$\begin{aligned} f_0(t) &= \sum_k f(k) L_m(t-k), \quad k \in \boldsymbol{Z} \qquad (7) \\ &= \begin{cases} \sum_k c_k^0 N_m(t-k) & m = m_e, \quad k \in \boldsymbol{Z} \\ \sum_k c_k^0 N_m(t + \frac{1}{2} - k) & m = m_o, \quad k \in \boldsymbol{Z} \end{cases} \end{aligned}$$

$$c_k^0 = \begin{cases} \sum_l f(l) \beta_{k+m/2-l}^m & m = m_e, \quad l \in \boldsymbol{Z} \\ \sum_l f(l) \beta_{k+(m-1)/2-l}^m & m = m_o, \quad l \in \boldsymbol{Z} \end{cases} \tag{8}$$

From Eq. (7) and Eq. (8), the following conclusions can be obtained. When m is m_e, the $f_0(x)$ becomes a standard form expressed as Eq. (4). However, when m is m_o, a half-sample-delay from the case when m is m_e occurs in c_k^0.

3. NEW DUAL-TREE WAVELET TRANSFORM USING RI-SPLINE WAVELET

3.1. The Construction of RI-Spline Wavelet

The RI-Spline wavelet, which is composed of a pair of Spline wavelets, can be described as the complex wavelets whose real component is the m_e Spline wavelets and whose imaginary component is the m_o Spline wavelets. This means that the real component is symmetric and the imaginary component is anti-symmetric.

We use the following notations:

$\psi_R(t)$ the real component of the RI-Spline wavelet
$\psi_I(t)$ the imaginary component of the RI-Spline wavelet
$N_R(t)$ the real component of the RI-Spline scaling function

$N_I((t)$ the imaginary component of the RI-Spline scaling function
$\psi_{m_e}(t)$ the m_e Spline wavelets
$\psi_{m_o}(t)$ the m_o Spline wavelets
$N_{m_e}(t)$ the m_e Spline scaling function
$N_{m_o}(t)$ the m_o Spline scaling function

Using these notations we define the RI-Spline wavelet and its scaling functions as follows:

$$\begin{aligned} &\psi(t) = \psi_R(t) + j\psi_I(t), \\ &\quad \psi_R(t) = (-1)^{(m_e-2)/2} ||\psi_{m_e}||^{-1} \psi_{m_e}(t + m_e - 1), \\ &\quad \psi_I(t) = (-1)^{(m_o+1)/2} ||\psi_{m_o}||^{-1} \psi_{m_o}(t + m_o - 1), \end{aligned} \tag{9}$$

$$\begin{aligned} N_R(t) &= N_{m_e}(t - m_e/2), \\ N_I(t) &= N_{m_o}(t - (m_o - 1)/2), \end{aligned} \tag{10}$$

where Eqs. (9) and (10) imply the phase adjustment. The normalization of the wavelets is conducted as follows:

$$\begin{aligned} \langle \psi_R, \psi_I \rangle &= 0, \\ ||\psi_R|| &= ||\psi_I|| = 1. \end{aligned} \tag{11}$$

Figure 1 shows an example of the RI-Spline wavelet, where Fig. 1(a) is the m=4,3 (m_e=4, m_o=3) RI-spline wavelet and (b) is its Fourier transform. From Fig. 1(b) it is clear that the real and imaginary components of the RI-Spline wavelet constitute an approximate Hilbert pair, which is valid for the case m=6,5, m=4,5 and m=6,7 and so on. Therefore, RI-Spline wavelet can be used as an alternate mother wavelet for the DTWT.

We denote the decomposition sequences of $\psi_R(t)$ as $\{a_k^R\}$ and $\{b_k^R\}$, and those of $\psi_I(t)$ as $\{a_k^I\}$ and $\{b_k^I\}$. We also denote the decomposition sequences of $\psi_{m_e}(t)$ as $\{a_k^{m_e}\}$ and $\{b_k^{m_e}\}$, and those of $\psi_{m_o}(t)$ as $\{a_k^{m_o}\}$ and $\{b_k^{m_o}\}$. Using these notations the decomposition sequences of the RI-Spline wavelet are expressed as follows:

$$\begin{aligned} a_k^R &= \sqrt{2} a_{k+m_e/2}^{m_e}, \\ b_k^R &= (-1)^{m_e/2+1} ||\psi_{m_e}|| \sqrt{2} b_{k+3m_e/2-2}^{m_e}, \end{aligned} \quad (12)$$

$$\begin{aligned} a_k^I &= \sqrt{2} a_{k+(m_o-1)/2}^{m_o}, \\ b_k^I &= (-1)^{(m_o+1)/2} ||\psi_{m_o}|| \sqrt{2} b_{k+3(m_o-1)/2}^{m_o}, \end{aligned} \quad (13)$$

We denote the reconstruction sequences of $\psi_R(t)$ as $\{p_k^R\}$ and $\{q_k^R\}$, and those of $\psi_I(t)$ as $\{p_k^I\}$ and $\{q_k^I\}$. We also denote the reconstruction sequences of the m_e Spline wavelet as $\{p_k^{m_e}\}$ and $\{q_k^{m_e}\}$, and those of the m=m_o Spline wavelet as $\{p_k^{m_o}\}$ and $\{q_k^{m_o}\}$. Using these notations the reconstruction sequences of the RI-Spline wavelet are expressed as follows:

$$\begin{aligned} p_k^R &= (\sqrt{2})^{-1} p_{k+m_e/2}^{m_e}, \\ q_k^R &= (-1)^{m_e/2+1} (||\psi_{m_e}|| \sqrt{2})^{-1} q_{k+3m_e/2-2}^{m_e}, \end{aligned} \quad (14)$$

$$\begin{aligned} p_k^I &= (\sqrt{2})^{-1} p_{k+(m_o-1)/2}^{m_o}, \\ q_k^I &= (-1)^{(m_o+1)/2} (||\psi_{m_o}|| \sqrt{2})^{-1} q_{k+3(m_o-1)/2}^{m_o}, \end{aligned} \quad (15)$$

In Eqs. (12), (13), (14) and (15), we omit the way of normalization of wavelets in each level.

3.2. Realizing a Half-Sample-Delay using Interpolation

As shown in Sec. 1, in the DTWT the filters must provide a half-sample-delay between the two trees. Fortunately, as shown in Sec.2, this half-sample-delay can be easily realized in the process of interpolation calculation when the m_e and m_o Spline scaling functions are used. However, the coefficient β_k^m in Eq. (6) is very difficult to calculate in the case m is m_o[5]. In order to calculate this coefficient, we propose a new synthetic-interpolation function, which is defined as follows:

$$N_s(t) = \sum_k K_k^R N_R(t-k) + \sum_k K_k^I N_I(t-k),\ k \in \boldsymbol{Z}, \quad (16)$$

In Eq.(16), it is necessary for $N_s(t)$ to be symmetric around the origin. It is also necessary for the energy of the input signal to be evenly shared in the real component $\sum K_k^R N_R(t-k)$ and the imaginary component $\sum K_k^I N_I(t-k)$ except near the Nyquist Frequency. The sequences K_k^R and K_k^I are designed so that they satisfy these conditions, and then the interpolation is computed as follows:

$$L_s(t) = \sum_k \beta_k^s N_s(t-k),\ L_s(k) = \delta_{k,0},\ k \in \boldsymbol{Z}. \quad (17)$$

By the sequence β_k^s satisfying Eq. (17), we have

$$f_0(t) = \sum_l c_l^0 N_s(t-l),\ c_l^0 = \sum_l f(l) \beta_{k-l}^s,\ l \in \boldsymbol{Z}, \quad (18)$$

and

$$c_{R,k}^0 = \sum_l c_l^0 K_{k-l}^R,\ c_{I,k}^0 = \sum_l c_l^0 K_{k-l}^I,\ l \in \boldsymbol{Z}, \quad (19)$$

where, $f_0(t)$ is the approximate input signal. Finally, we obtain the interpolation as follows:

$$f_0(t) = \sum_k c_{R,k}^0 N_R(t-k) + \sum_k c_{I,k}^0 N_I(t-k),\ k \in \boldsymbol{Z} \quad (20)$$

From Eq. (20), it is clear that both $\sum_k c_{R,k}^0 N_R(t-k)$ and $\sum_k c_{I,k}^0 N_I(t-k)$ terms of Eq. (20) become the standard forms expressed as the Eq. (4).

3.3. Coherent Dual-Tree Algorithm

We propose the coherent Dual-Tree Algorithm shown as Fig. 2. In this algorithm, the real sequences $c_{R,k}^0$ and the imaginary sequences $c_{I,k}^0$ are first calculated from $f_0(t)$ by the interpolation expressed as Eqs. (18) and (19). Then they are decomposed ordinarily by Eqs. (21) and (22):

$$c_{R,k}^{j-1} = \sum_l a_{l-2k}^R c_{R,l}^j,\quad d_{R,k}^{j-1} = \sum_l b_{l-2k}^R c_{R,l}^j, l \in \boldsymbol{Z} \quad (21)$$

$$c_{I,k}^{j-1} = \sum_l a_{l-2k}^I c_{I,l}^j,\quad d_{I,k}^{j-1} = \sum_l b_{l-2k}^I c_{I,l}^j,\quad l \in \boldsymbol{Z} \quad (22)$$

For reconstruction, the reconstruction tree shown in Fig. 2(b) can be applied. The inverse transformation can be calculated by the next equations:

$$c_{R,k}^j = \sum_l (p_{k-2l}^R c_{R,l}^{j-1} + q_{k-2l}^R d_{R,l}^{j-1}),\quad l \in \boldsymbol{Z} \quad (23)$$

$$c_{I,k}^j = \sum_l (p_{k-2l}^I c_{I,l}^{j-1} + q_{k-2l}^I d_{I,l}^{j-1}),\quad l \in \boldsymbol{Z} \quad (24)$$

By Eq. (11), we have

$$\begin{aligned} &\langle \psi_{R,k}^j, \psi_{I,k}^j \rangle = 0, \\ &||\psi_{R,k}^j|| = ||\psi_{I,k}^j|| = 1, \end{aligned} \quad (25)$$

The norm of the synthetic wavelet can be computed as follows:

$$||d_{R,k}^j \psi_{R,k}^j + d_{I,k}^j \psi_{I,k}^j|| = \sqrt{(d_{R,k}^j)^2 + (d_{I,k}^j)^2} \quad (26)$$

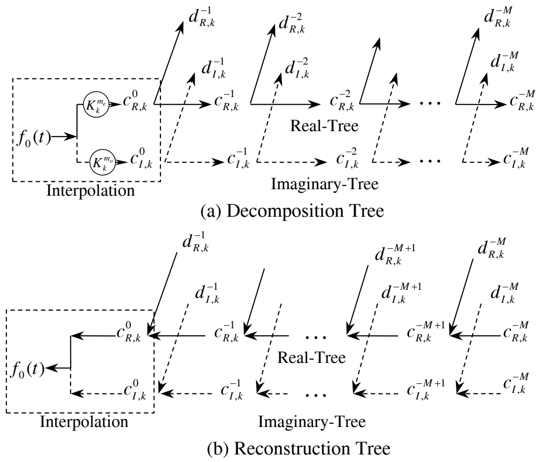


Fig. 2. New Dual–Tree Algorithm.

As shown above, our Dual-Tree Algorithm is very simple and it is not necessary to provide the delay of one tree's filter which is one sample offset from another tree's filter in level -1. Therefore, complex analysis can be carried out coherently in all analysis levels.

4. VERIFYING TRANSLATION INVARIANCE BY EXPERIMENTS

Figure 3 shows the change of the energy in each level that is obtained by shifting one sample of an impulse signal's position, where (a) is obtained by our DTWT using the RI-Spline wavelet, (b) and (c) are obtained by the current DWT using Daubechise 8 (D8) and m=4 Spline (S4) wavelets. As is shown in Fig. 3, the energy in each level by using the RI-Spline wavelet is very stable, and almost is not influenced by shifting. By comparison with RI-Spline wavelet, the energy obtained by the S4 and D8 wavelets are greatly changed. From these experiments, we can conclude that our DTWT using the RI-Spline wavelet solves the translation variance problem of DWT.

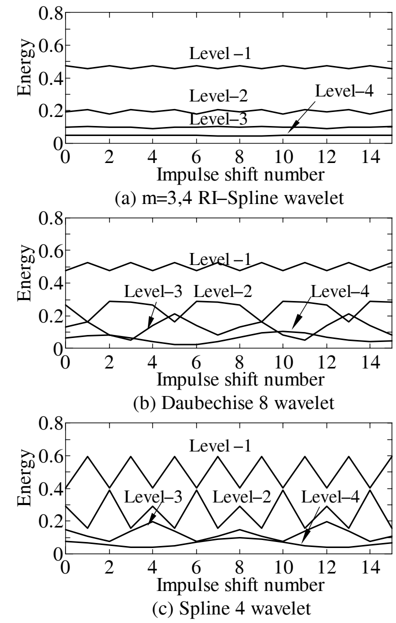


Fig. 3. Change of the impulse's Energy obtained by different impulse position.

5. CONCLUSIONS

In this study, we proposed a new complex wavelet, the DTWT using the RI-spline wavelet, for which we also proposed a coherent Dual–Tree Algorithm using the interpolation to provide a half-sample-delay between the two trees. Finally we experimentally showed that translation invariance can be obtained by our method. The results obtained can be summarized as follows:

1)A new complex wavelet, the RI- Spline Wavelet which is constructed simply by two spline wavelets which constitute an approximate Hilbert pair, was proposed. It is useful for solving the translation variance problem of the DWT, without so increasing the computational cost much. The RI-spline wavelet is simple and we do not need a complicated process for designing new filters for DTWT.

2)The interpolation method, which includes the calculation of coefficients c_k^0 from the signal $f(t)$, is useful to realize a half-sample-delay between the two trees. Therefore, complex analysis can be carried out coherently from level -1.

6. REFERENCES

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