Design of Approximate Hilbert Transform Pair of Wavelets with Exact Symmetry

David B. H. Tay*, Marimuthu Palaniswami†

*Department of Electronic Engineering, LaTrobe University, Victoria 3086, Australia.
†Department of Electrical and Electronic Engineering, The University of Melbourne
Victoria 3010, Australia

Abstract— **This paper presents a new technique for designing pairs of filter banks whose corresponding wavelet function are approximate Hilbert transform of each other. The filters have exact linear phase which yields biorthogonal wavelets with exact symmetry. The technique is based on *matching* the frequency response of a given odd-length filter bank with an even-length filter bank. The class of EBFB (Even-length Bernstein Filter Bank) is utilized in the matching design. The EBFB has perfect reconstruction and vanishing moments properties structurally imposed and this simplifies the design process. The design is achieved through a non-iterative least squares method.**

I. Introduction

It is almost two decades since the pioneering work of Daubechies and Mallat showed the relationship between multirate filter banks and wavelet. They introduced the dyadic DWT (Discrete Wavelet Transform) which has served the engineering and scientific community well as an important signal processing tool in various applications. The testament to its success is seen in its adoption in international standards like the JPEG2000 for image compression. The DWT however is not without its disadvantages and limitations and this has lead to the development of transforms that are extensions or variants of the DWT. One of the main disadvantage of the DWT is its shift-variant property and this is due to the inherent multirate nature of the transform. This has lead some researchers to develop over-complete transforms to reduce the degree of shift-variancy.

Recently Selesnick [1],[2] introduced the notion of a pair of filter banks where the equivalent wavelet of one bank is the Hilbert transform of the wavelet of the other bank (and will be referred to as a Hilbert-Pair). The use of a Hilbert-Pair for transient detection was proposed by Abry and Flandrin [3] and for waveform encoding by Ozturk [4]. The dual-tree complex wavelets of Kingsbury [5] which has approximate shift-invariance is also related to the Hilbert-Pair.

The design of approximate Hilbert-Pairs was addressed in [1],[2]. Only orthogonal wavelets were considered in [1] but both orthogonal and biorthogonal wavelets were considered in [2]. None of the wavelets in [1] and [2] however have exact symmetry although it was advocated that symmetry is desirable for better directional selectivity [2]. Some of the biorthogonal wavelets in [2] however have approximate symmetry but exact symmetry is unattainable using the design technique proposed therein. In this work we present an alternate technique that utilizes the EBFB (Even-length Bernstein Filter Bank) [6] and yields exactly symmetric wavelets. The technique is simple to apply and essentially involves solving a linear least squares problem.

II. Hilbert Transform Wavelet Pairs

A standard 2-channel filter bank is defined by four filters:
1. Low-pass: $H_0(z)$ (analysis) and $F_0(z)$ (synthesis).
2. High-pass: $H_1(z)$ (analysis) and $F_1(z)$ (synthesis).
Perfect reconstruction is achieved if the following equations are satisfied: (i) $H_1(z) = z^{-1}H_0(-z)$; (ii) $F_1(z) = zF_0(-z)$ and (iii) $H_0(z)F_0(z) + H_0(-z)F_0(-z) = 1$. By denoting the coefficients of the filters by lower case letters; eg. $f_0(n)$ for $F_0(z)$, ie. $F_0(z) = \sum_n f_0(n)z^{-n}$; the synthesis scaling function $\phi(t)$ and synthesis wavelet $\psi(t)$ are given implicitly by the following two-scale equations:

$$\phi(t) = \frac{2}{F_0(1)} \sum_k f_0(k)\, \phi(2t-k)$$
$$\psi(t) = \frac{2}{F_1(1)} \sum_k f_1(k)\, \phi(2t-k)$$

Similar equations exist relating the analysis scaling function $\tilde{\phi}(t)$ and the analysis wavelet $\tilde{\psi}(t)$ to the coefficients of the analysis filters.

A Hilbert Pair is made up of two filter banks that are related to each other. The filters in one bank is denoted by the superscript h, ie. $(H_0^h, H_1^h, F_0^h, F_1^h)$, and the other banks' filter is denoted by the superscript g, ie. $(H_0^g, H_1^g, F_0^g, F_1^g)$. The corresponding scaling functions and wavelets are also denoted by the same superscripts, eg. $\psi^h(t)$ for bank h and $\psi^g(t)$ for bank g. Suppose it is required that $\psi^g(t)$ is the Hilbert transform of $\psi^h(t)$, ie.

$$\Psi^g(\omega) = \begin{cases} -j\Psi^h(\omega) & \text{for} \quad \omega > 0 \\ j\Psi^h(\omega) & \text{for} \quad \omega < 0 \end{cases} \qquad (1)$$

where $\Psi^h(\omega)$ and $\Psi^g(\omega)$ are the Fourier transform of $\psi^h(t)$ and $\psi^g(t)$ respectively. It was shown by Selesnick [1] that a sufficient condition for (1) is a relationship between the low-pass filters of the two banks:

$$F_0^g(\omega) = e^{-j\omega/2} F_0^h(\omega) \quad (2)$$

(Note the slight abuse of notation, ie. $F_0^g(\omega) \equiv F_0^g(e^{j\omega})$). Condition (2) was shown recently in [7] to be also necessary. Note that equations (1) and (2) are for the synthesis side of the banks and there are similar equations at the analysis side.

Equation (2) implies that the impulse response $f_0^g(n)$ is a half-sampled delayed version of $f_0^h(n)$. This cannot be satisfied exactly using FIR filters so an approximation, ie. $F_0^g(\omega) \approx e^{-j\omega/2} F_0^h(\omega)$, is used in actual designs. Previous design approaches are:

1. Minimize the upsampled error $E(\omega) \equiv F_0^g(2\omega) - e^{-j\omega} F_0^h(2\omega)$ at the vicinity of $\omega = 0$ [1]. Specifically $E(\omega)$ and a certain prescribed number of its derivatives at $\omega = 0$ are set to zero. The technique requires the solution of non-linear equations and was achieved using Grobner bases in [1].
2. Using an allpass filter that approximates a half-sample delay, ie. $A(z) \approx z^{-1/2}$, to assist in the construction of the filters [2]. The filters constructed are related as follows:

$$F_0^g(\omega) = A(\omega) F_0^h(\omega) \quad (3)$$

The technique in [2] is essentially a spectral factorization approach and no explicit frequency domain criterion is imposed in the designs.

III. Matching Technique of Design

Equation (2) is a complex valued equation and with the technique in [2], (ie. using (3)), the magnitude part of the equation is exactly satisfied, ie. $|F_0^g(\omega)| = |F_0^h(\omega)|$. The approximation is then in the phase and the approximation quality will depend on the type of all-pass filter $A(z)$ used.

In this paper a complementary approach is adopted which is described next. The filter bank h comprise of odd-length linear phase filters and bank g comprise of even-length linear phase filters. For convenience the center of symmetries of the filters' impulse response is chosen to be as close as possible to the origin. The filters frequency response can then by written as [8]:

$$F_0^h(\omega) = f_0(0) + 2\sum_{n\neq 0} f_0^h(n)\cos(n\omega) \quad (4)$$

$$\begin{aligned} F_0^g(\omega) &= e^{-j\omega/2} F_{0,R}^g(\omega) \\ &\equiv e^{-j\omega/2}\, 2\sum_n f_0^g(n)\cos((n-\tfrac{1}{2})\omega) \quad (5) \end{aligned}$$

Both $F_0^h(\omega)$ and $F_{0,R}^g(\omega)$ are real valued functions and are essentially the magnitude response of the filters. With (4) and (5) its readily seen that $\angle F_0^g(\omega) = -\frac{\omega}{2} + \angle F_0^h(\omega)$ (provided $F_0^h(\omega)$ and $F_{0,R}^g(\omega)$ have the same sign), ie. the phase part of (2) is exactly satisfied. The approximation is then in the amplitude, ie.

$$F_{0,R}^g(\omega) \approx F_0^h(\omega) \quad (6)$$

To design a Hilbert pair the odd-length filter bank h is chosen or constructed first. The even-length filter in bank g is then designed to *match* the filter in bank h, ie. eqn. (6). In this paper the class of EBFB (Even-length Bernstein Filter Bank) is used for bank g and is briefly discuss next (more details in [6]).

A. Even Length Bernstein Filter Bank

The key component to the EBFB is the Parametric Bernstein Polynomial which is defined as:

$$B_N(x;\alpha) \equiv \sum_{i=0}^{N} f(i) \begin{pmatrix} N \\ i \end{pmatrix} x^i (1-x)^{N-i} \quad (7)$$

where N is odd, $\alpha = [\ \alpha_0 \ \ \dots \ \ \alpha_{(N-1)/2}\ ]^T$ and

$$f(i) \equiv \begin{cases} 1-\alpha_i & 0 \le i \le \frac{1}{2}(N-1) \\ \alpha_{N-i} & \frac{1}{2}(N+1) \le i \le N \end{cases}$$

First the following two Bernstein polynomial with different parameters are defined: $B_1(x) \equiv B_N(x;\alpha/\alpha_0 = 0)$ and $B_2(x) \equiv B_M(x;\beta)$. Next the following functions are defined:

$$\begin{aligned} H(x) &\equiv (1-x)^{-1/2} B_1(x) \\ F(x) &\equiv (1-x)^{1/2}\ [\ B_1(x)\ +\ 2\,B_2(x) \\ &\quad -\ 2\,B_1(x)B_2(x)\ ] \end{aligned}$$

Finally the low-pass filters of the EBFB are given by

$$H_0^g(z) \equiv z^{1/2} H(\tfrac{1}{4} z (1-z^{-1})^2) \quad (8)$$

$$F_0^g(z) \equiv z^{-1/2} F(\tfrac{1}{4} z (1-z^{-1})^2) \quad (9)$$

The advantages of the EBFB are that perfect reconstruction and vanishing moments are structurally imposed. The desired degree of vanishing moments for $H_0^g(z)$, $(2L_H + 1)$, can be achieved by setting $\alpha_i = 0$ for $i = 0, \dots, L_H$. The degree of vanishing moments for $F_0^g(z)$ is $(2\ \min(L_H, L_F) + 3)$ where $\beta_i = 0$ for $i = 0, \dots, L_F$. The optimization that needs to be carried out on the remaining non-zero free parameters is an unconstrained one.

To facilitate the design process the following change of variable

$$y = (1-x)^{1/2} = \cos\left(\frac{\omega}{2}\right) \quad (10)$$

is applied to the filter functions [6]. Using the variable y, the filter function $H_0^g(z)$ can be written as [6]:

$$H^g(y) = K^H(y) - \sum_l k_l^H(y)\,\alpha_l$$

where $K^H(y)$ and $k_l^H(y)$ are polynomial functions in y and are independent of α_l; and the filter function $F_0^g(z)$ can be written as:

$$F^g(y) = K^F(y) - \sum_l k_l^F(y)\,\beta_l$$

where $K^F(y)$ and $k_l^F(y)$ are polynomial in y and are independent of β_l but linearly dependent on α_l. Expressions for K^H, k_l^H, K^F and k_l^F are available in [6].

B. Least Squares Design

Once bank h has been determined, apply the change of variable (10) to its low-pass filters: $H_0^h(\omega) \to H^h(y)$ and $F_0^h(\omega) \to F^h(y)$. Note that $H^h(y)$ and $F^h(y)$ are polynomials in y. To match filter H_0^g to H_0^h the following objective function is used:

$$\begin{aligned} E &= \int_0^1 (H^g(y) - H^h(y))^2 \, dy \\ &= \int_0^1 \left(K^H(y) - H^h(y) - \sum_l k_l^H(y)\, \alpha_l \right)^2 dy \end{aligned}$$

As can be clearly seen E is a quadratic function of the free parameters α_l and its minimization requires the solution of simple linear equations. The coefficients of the linear equations can be computed analytically because all functions involved are polynomial. Once the α_l have been obtained, the β_l can be obtained using a similar procedure by matching F_0^g to F_0^h.

IV. Design Examples

Two well known odd-length filter banks are considered here and the design of the corresponding matching even-length filter bank is presented.

Example 1: the odd-length filter bank is the Daubechies 5/7 pair [9],[10] which is almost orthogonal. For the EBFB, $N = 3$ and $M = 1$, and this yields filters of length 6 and 10. We choose minimum values for $L_H = 0$ ($\alpha_0 = 0$) and $L_F = -1$ and this gives one zero (at $z = -1$) each for the length 6 and 10 filters. The length 6 (10) filter is matched to the length 5 (7) filter. The optimized values of the free parameters are

$$\alpha_1 = 0.1385, \quad \beta_1 = 0.1501$$

Figure 1 shows the wavelets from the odd- and even-length filter banks. Figure 2 shows the spectrums of $\psi^h(t)$ and $(\psi^h(t) + j\psi^g(t))$.

Example 2: the odd-length filter bank is the celebrated Daubechies 9/7 pair [9],[10] which is employed in the JPEG2000 standard. For the EBFB, $N = M = 5$, and this yields filters of length 10 and 22. We choose $L_H = 0$ ($\alpha_0 = 0$) and $L_F = 0$ ($\beta_0 = 0$) and this gives one zero (at $z = -1$) for the length 10 filter and 3 zeros for the length 22 filter. The length 10 (22) filter is matched to the length 7 (9) filter. The optimized values of the free parameters are

$$\alpha_1 = 0.0025, \;\; \alpha_2 = 0.0253, \;\; \beta_1 = 0.1005, \;\; \beta_2 = 1.0811$$

Figure 3 shows the wavelets from the odd- and even-length filter banks. Figure 4 shows the spectrums of $\psi^h(t)$ and $(\psi^h(t) + j\psi^g(t))$.

A. Discussion

It is clearly seen that all the wavelets are symmetrical and each Hilbert pair of wavelets have opposite types of symmetry, ie. mirror-symmetric / anti-symmetric. In comparison the almost symmetric pair of wavelets in [2] have only one type of symmetry, namely mirror-symmetric.

The spectrum $|\Psi^h(\omega) + j\Psi^g(\omega)|$ is essentially one-sided and matches well the spectrum $|\Psi^h(\omega)|$ for $\omega > 0$. Plots of $\angle \frac{\Psi^g}{\Psi^h}$ for both examples (not shown here) verify that the phase part of eqn. (1) is satisfied, ie. $\angle \frac{\Psi^g}{\Psi^h}(\omega) = -j \,\text{sign}(\omega)$, except for a small number of frequency points. This means that $(\psi^h(t) + j\psi^g(t))$ is an approximate analytic version of $\psi^h(t)$ and shows that a reasonably good approximation to the Hilbert transform is achieved.

V. Summary

The design of wavelet pairs that are related through the Hilbert transform has been presented. The wavelets are biorthogonal and have exact symmetry. The wavelets in each Hilbert pair have opposite symmetries. The design is achieved by matching an even-length filter bank to a given odd-length filter bank. The Even-length Bernstein Filter Bank was utilized for the even bank and this simplified the design process. A least squares formulation was proposed for the design and this required the solution of simple linear equations. Examples were presented which showed that approximate Hilbert pairs can be designed with ease.

Further research in this direction includes:
1. Using a different error norm (to the error squared norm) for the optimization, eg. minimax error.
2. Matching an odd-length filter bank to a predetermined even-length filter bank.

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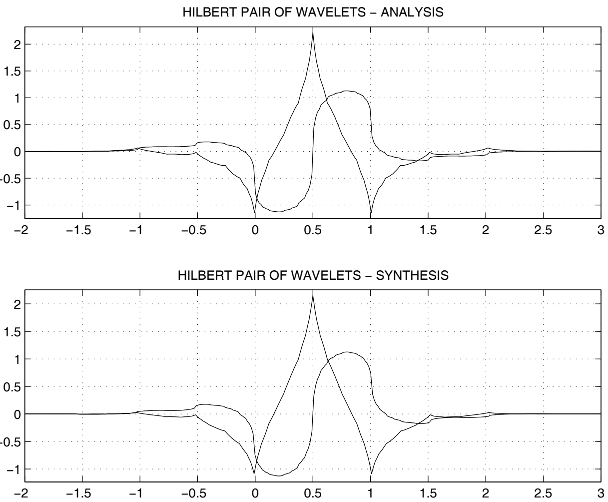

Fig. 1. Wavelets from example 1. Mirror-symmetric wavelets from odd-length filter bank and anti-symmetric wavelets from even-length filter bank.

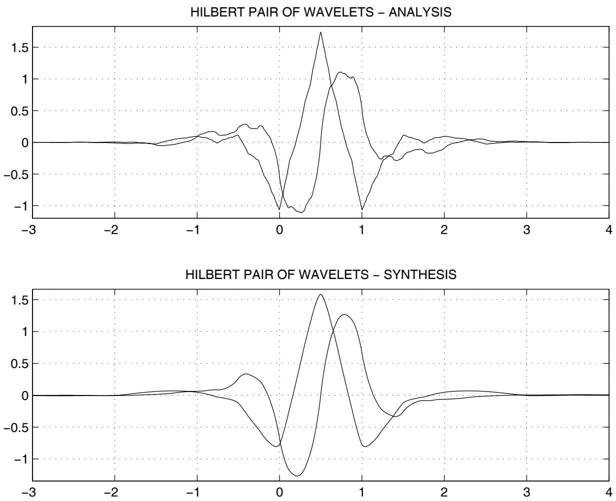

Fig. 3. Wavelets from example 2. Mirror-symmetric wavelets from odd-length filter bank and anti-symmetric wavelets from even-length filter bank.

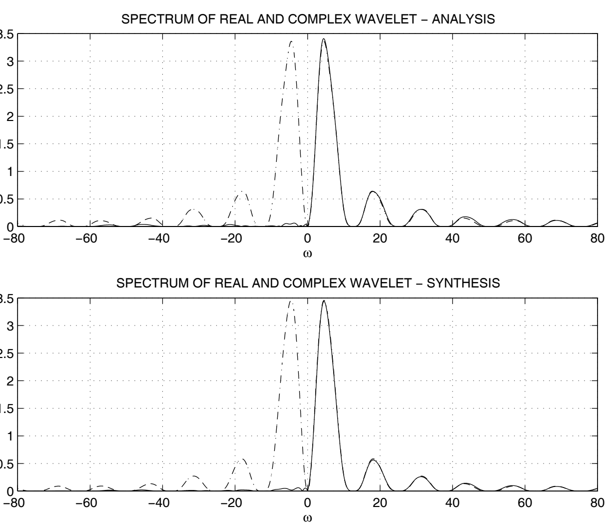

Fig. 2. Wavelet spectrum from example 1. Dotted line - real wavelet from odd-length filter bank (multiplied by 2), ie. $2|\Psi^h(\omega)|$. Solid line - approximate complex analytic wavelet, ie. $|\Psi^h(\omega) + j\Psi^g(\omega)|$

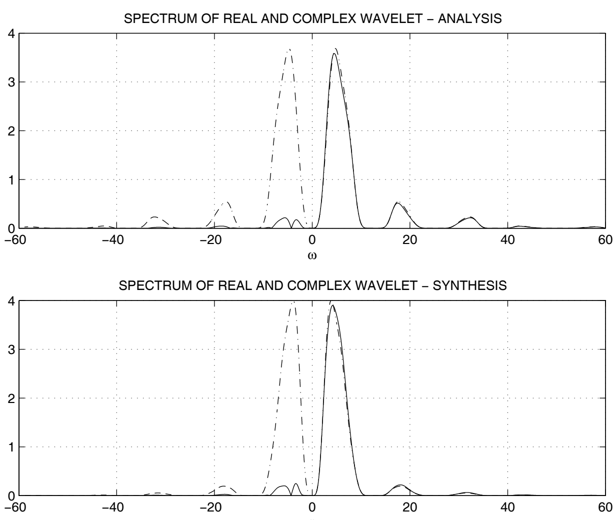

Fig. 4. Wavelet spectrum from example 2. Dotted line - real wavelet from odd-length filter bank (multiplied by 2), ie. $2|\Psi^h(\omega)|$. Solid line - approximate complex analytic wavelet, ie. $|\Psi^h(\omega) + j\Psi^g(\omega)|$