WEIGHTED MEDIAN BASED FILTERS FOR THE COMPLEX DOMAIN

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ABSTRACT

Weighted median (WM) filtering structures for complex-valued samples have been proposed but none of them allows the use of complex-valued weights. This paper defines complex-valued weighting for median filters with complex-valued input samples. Two different approaches to the handling of weights in the complex domain are presented, both derived from characteristics of complex-valued linear filters, resulting in two definitions of the complex weighted median filter. The LMS optimizations of the proposed filtering schemes are also presented. Simulations are shown illustrating the performance of the new complex WM filter structures compared with previous approaches to the problem and with classical linear filters.

1. INTRODUCTION

Although robust signal processing methods for real-valued data have been investigated extensively in the past decade, approaches for complex-valued signals have not received attention. This is the case even for the well known weighted medians. If weighting of the complex samples is desirable, the existing definitions are severely limited in that only positive weights are allowed.

Several approaches to overcome the computational complexity of complex valued medians have been proposed [1, 2]. However, none of these structures allows complex valued weights. This paper defines complex weighting in WM filters based on the concepts of phase-coupling and real-imaginary coupling. In order to introduce these concepts it is useful to formulate this filtering problem from its statistical roots.

Under the Laplacian model, the maximum likelihood estimate of location is given by [3]

$$\hat{\beta} = \arg\inf_{\beta} \sum_{i=1}^{N} W_i |X_i - \beta|, \tag{1}$$

where the input samples $X_i|_{i=1}^{N}$ are complex-valued. There is no closed-form solution to (1). The suboptimal approach introduced by Astola [4], the marginal complex median, is a fast approximation that considers real and imaginary parts independent, allowing to break up the complex-valued optimization into two real-valued optimizations. This approach admits only positive weights. To overcome these limitations, we introduce the concept of *phase coupling*, consisting in decoupling the phase of the complex-valued weight and merging it to the associated complex-valued input sample, and use it to define the *phase coupled* complex WM filter. A second approach to the weighting concept that exploits the correlation between real and imaginary parts of the input samples is introduced. Based on this concept we define the *real-imaginary coupled* complex WM. This structures were initially introduced in [5, 6] but a more conceptual approach is presented here.

2. COMPLEX WEIGHTED MEDIAN FILTERS

The weighting strategy is essential to filtering operations. Many communications related applications require filtering structures admitting complex-valued weights. For linear filters, there are no difficulties in obtaining the optimal weights. However, due to the nonlinear nature of the median operation, the optimal complex weight design for median-type filters has not been explored in the literature to our best knowledge, and even the meaning of complex weighting itself is vague. In this section we propose a set of complex weighted median filter structures that can fully exploit the power of complex weighting and still keep the advantages inherited from univariate medians. The simplest approach to attain complex WM filtering is to perform marginal operations where the real component of the weights $W_{R_i}|_{i=1}^{N}$ affect the real part of the samples $X_{R_i}|_{i=1}^{N}$ and the imaginary component of the weights $W_{I_i}|_{i=1}^{N}$ affect the imaginary part of the samples $X_{I_i}|_{i=1}^{N}$. This approach referred to as *marginal* complex WM filter outputs:

$$\begin{aligned}\hat{\beta}_{marginal} = &\text{MEDIAN}\left(|W_{R_i}|\diamond \text{sgn}(W_{R_i})X_{R_i}|_{i=1}^{N}\right)\\ &+j\text{MEDIAN}\left(|W_{I_i}|\diamond \text{sgn}(W_{I_i})X_{I_i}|_{i=1}^{N}\right),\end{aligned} \tag{2}$$

where $\diamond$ is the replication operator defined as $W \diamond X = \overbrace{X, X, \cdots, X}^{W times}$ and the real and imaginary components are decoupled [1]. The definition in (2) assumes that the real and imaginary components of the input samples are independent. On the other hand, if the real and imaginary domains are correlated, better performance is attainable by mutually *coupling* the components of the signal and weights.

Consider the mean operation with complex-valued weights $W_i = |W_i| e^{j\theta_i}|_{i=1}^{N}$,

$$\begin{aligned} \bar{\beta} &= \text{MEAN}\left(W_1^* X_1, \cdots, W_N^* X_N\right) \\ &= \frac{1}{N} \sum_{i=1}^{N} |W_i| \cdot e^{-j\theta_i} X_i. \end{aligned} \tag{3}$$

The simple manipulation in (3) reveals that the weights have two roles in the weighted mean operation. First their phases are coupled into the samples changing them into a new group of *phased* samples, and then the magnitudes of the weights are applied. The process of decoupling the phase from the weight and merge it to the associated input sample is called *phase coupling*. The definition of the *phase coupled* complex WM filter follows by analogy.

2.1. Phase Coupled Complex WM Filter

Given the complex valued samples $X_1, X_2, ... X_N$ and the complex valued weights $W_i = |W_i| e^{j\theta_i}, i = 1, .., N$, the output of the phase coupled complex WM is defined as

$$\hat{\beta} = \arg\inf_{\beta} \sum_{i=1}^{N} |W_i| |e^{-j\theta_i} X_i - \beta|. \tag{4}$$

This definition of the complex weighted median delivers a rich class of complex median filtering structures. The solution to (4), however, suffers from computational complexity as the cost function must be searched for its minimum. Any one of the already mentioned suboptimal approximations, i.e. assume that the output $\hat{\beta}$ is one of the phase-coupled input samples or, split the problem into real and imaginary parts, arise as effective ways to reduce the complexity. The first approximation referred to as the *selection phase coupled* complex WM reduces (4) to,

$$\hat{\beta} = \arg\min_{\beta \in \{e^{-j\theta_i} X_i\}} \sum_{i=1}^{N} |W_i| |e^{-j\theta_i} X_i - \beta|. \tag{5}$$

Since $\hat{\beta}$ is confined to be one of the phase-coupled inputs, this filter is intuitively desirable when very sensitive information is carried by the phase of samples and weights. However the computation of (5) requires the evaluation of the cost function for each one of the phase-coupled input samples and thus may be not suitable for fast applications. The following definitions provide efficient and fast complex-valued WM filter structures.

[1] For a method for the calculation of the weighted median with non-integer weights refer to [3].

2.2. Marginal Phase Coupled Complex Weighted Median Filter

The *marginal phase coupled* complex WM filter reduces the output in (4) to the following two real-valued weighted medians,

$$\begin{aligned} \hat{\beta}_R &= \arg\inf_{\beta_R} \sum_{i=1}^{N} |W_i| |\text{Re}\{e^{-j\theta_i} X_i\} - \beta_R| \\ &= \text{MEDIAN}(|W_i| \diamond \text{Re}\{e^{-j\theta_i} X_i\} \,|_{i=1}^{N}), \end{aligned} \tag{6}$$

$$\begin{aligned} \hat{\beta}_I &= \arg\inf_{\beta_I} \sum_{i=1}^{N} |W_i| |\text{Im}\{e^{-j\theta_i} X_i\} - \beta_I| \\ &= \text{MEDIAN}(|W_i| \diamond \text{Im}\{e^{-j\theta_i} X_i\} \,|_{i=1}^{N}), \end{aligned} \tag{7}$$

where $\text{Re}\{\cdot\}$ and $\text{Im}\{\cdot\}$ denote real and imaginary part respectively, and the filter output is $\hat{\beta} = \hat{\beta}_R + j\hat{\beta}_I$.

2.3. Real-Imaginary Coupled Complex Weighted Median Filter

In phase coupling, the phase of the weights modify the phase of the input samples and the norms of the weights perform the smoothing operation in the real and imaginary domains independently. As an alternative, we can use the complex-valued weights to exploit the coupling characteristics between the real and imaginary parts of the input signal. These characteristics are shown in the computation of the complex linear filter. Letting $\mathbf{W} = [W_1, W_2, ..., W_N]^T$ and $\mathbf{X} = [X_1, X_2, ..., X_N]^T$, the output of the complex linear filter is

$$\begin{aligned} \bar{\beta} &= \mathbf{W}^H \mathbf{X} = \left(\mathbf{W}_R^T - j\mathbf{W}_I^T\right) \cdot (\mathbf{X}_R + j\mathbf{X}_I) \\ &= \left(\mathbf{W}_R^T \mathbf{X}_R + \mathbf{W}_I^T \mathbf{X}_I + j\mathbf{W}_R^T \mathbf{X}_I - j\mathbf{W}_I^T \mathbf{X}_R\right). \end{aligned} \tag{8}$$

where $\mathbf{W}_R = \text{Re}\{\mathbf{W}\}$, $\mathbf{W}_I = \text{Im}\{\mathbf{W}\}$, $\mathbf{X}_I = \text{Im}\{\mathbf{X}\}$, $\mathbf{X}_R = \text{Re}\{\mathbf{X}\}$. Defining the following vectors: $\mathbf{W}_{RI}^T = \left[\mathbf{W}_R^T \mid \mathbf{W}_I^T\right]$, $\mathbf{X}_{RI}^T = \left[\mathbf{X}_R^T \mid \mathbf{X}_I^T\right]$, $\mathbf{X}_{IR-}^T = \left[\mathbf{X}_I^T \mid -\mathbf{X}_R^T\right]$, the complex linear filtering structure can be rewritten as,

$$\bar{\beta} = \mathbf{W}^H \mathbf{X} = \mathbf{W}_{RI}^T \mathbf{X}_{RI} + j\mathbf{W}_{RI}^T \mathbf{X}_{IR-}. \tag{9}$$

Thus the complex linear filter can be split into two real linear filtering structures. This representation is very convenient since the vector $\mathbf{W}_{RI}^T$ is used in both real-valued filtering operations. Additionally, the two data vectors $\mathbf{X}_{RI}$ and $\mathbf{X}_{IR-}$ are created from the original input data vector which is not computationally expensive. The linear complex filtering representation in (9) motivates the following definition, namely the *real-imaginary coupled* complex WM,

$$\begin{aligned} \hat{\beta} &= \text{MEDIAN}(\mathbf{W}_{RI}^T \diamond \mathbf{X}_{RI}) + j\text{MEDIAN}(\mathbf{W}_{RI}^T \diamond \mathbf{X}_{IR-}) \\ &= \text{MEDIAN}(|W_{R_i}| \diamond \text{sgn}(W_{R_i}) X_{R_i}|_{i=1}^{N}, \\ &\qquad |W_{I_i}| \diamond \text{sgn}(W_{I_i}) X_{I_i}|_{i=1}^{N}) \\ &\quad + j\,\text{MEDIAN}(|W_{R_i}| \diamond \text{sgn}(W_{R_i}) X_{I_i}|_{i=1}^{N}, \\ &\qquad |W_{I_i}| \diamond - \text{sgn}(W_{I_i}) X_{R_i}|_{i=1}^{N}) \end{aligned} \tag{10}$$

where $W_{R_i} = \mathrm{Re}\{W_i\}$, $W_{I_i} = \mathrm{Im}\{W_i\}$, $X_{R_i} = \mathrm{Re}\{X_i\}$ and $X_{I_i} = \mathrm{Im}\{X_i\}$ for $i = 1, 2, ..., N$.

Three important characteristics provide a strong support for this definition. First, the sample vectors $\mathbf{X}_{RI}$ and $\mathbf{X}_{IR^-}$ are intrinsically coupled in the complex space. This facilitates to perform the optimization of the filter in a joint (real and imaginary components) manner. Second, the computation in (10) does not require any suboptimal implementation since it consists of two real-valued weighted median operations that are known to be fast. Lastly, the definition in (10) reduces to the real-valued weighted median filter when the samples and weights are real-valued.

3. OPTIMIZATION

In this section, adaptive algorithms for both marginal phase coupled and real-imaginary coupled complex weighted median filters in the minimum MSE sense are shown.

3.1. Optimal Marginal Phase Coupled Complex WM

Given the complex-valued samples $\{X_i \mid_{i=1}^N\}$, the complex-valued weights $\{|W_i|e^{-j\theta_i} \mid_{i=1}^N\}$, define $P_i = e^{-j\theta_i} X_i \mid_{i=1}^N$ as the phase-coupled input samples and its real and imaginary parts as $P_{R_i} = \mathrm{Re}\{P_i\}$, $P_{I_i} = \mathrm{Im}\{P_i\}$.

Assume the observed process$\{X(n)\}$ and the desired process $\{\beta(n)\}$ are jointly stationary. The filter output $\hat{\beta}(n)$ estimating the desired signal $\beta(n)$ is given in (6) and (7). The cost function to minimize is

$$\begin{aligned} J(n) &= \mathrm{E}\{|\beta(n) - \hat{\beta}(n)|^2\} \\ &= \frac{1}{4}\mathrm{E}\left\{\left(\int_{-\infty}^{\infty} e_R^s \, \mathrm{d}s\right)^2 + \left(\int_{-\infty}^{\infty} e_I^r \, \mathrm{d}r\right)^2\right\}, \quad (11) \end{aligned}$$

where $e_R = \mathrm{Re}\{\beta(n)\} - \hat{\beta}_R(n)$, $e_I = \mathrm{Im}\{\beta(n)\} - \hat{\beta}_I(n)$, $\hat{\beta}_R(n) = \mathrm{Re}\{\beta(n)\}$, $\hat{\beta}_I(n) = \mathrm{Im}\{\beta(n)\}$.

It can be shown that the minimization of (11) leads to the following LMS weight update equation:

$$\begin{aligned} W_i(n+1) &= W_i(n) + \mu\{-\nabla J(n)\} \\ &\approx W_i(n) + \mu e^{j\theta_i}\Big\{e_R(n)\mathrm{sgn}(P_{R_i}(n) - \hat{\beta}_R(n)) \\ &\quad + e_I(n)\mathrm{sgn}(P_{I_i}(n) - \hat{\beta}_I(n)) \qquad (12) \\ &\quad + 2je_R(n)(P_{I_i}(n)\delta(P_{R_i}(n) - \hat{\beta}_R(n)) \\ &\quad + 2je_I(n)(P_{R_i}(n)\delta(P_{I_i}(n) - \hat{\beta}_I(n))\Big\}. \end{aligned}$$

3.2. Optimal Real-Imaginary Coupled Complex WM

It is desired to find the optimum set of coupled real-valued weights $\mathbf{W}_{RI}$ needed in the *real-imaginary coupled* complex WM in the minimum squared error sense. A LMS algorithm for the adaptation of the weights is developed where the cost function to minimize will be the same as in the previous case. The desired LMS recursive equation is found to be as follows

$$\begin{aligned} W_{RI_i}(n+1) &= W_{RI_k}(n) + \mu\{-\nabla J(n)\} \qquad (13) \\ &= W_{RI_i}(n) + \mu\Big[e_R(n)\mathrm{sgn}(W_{RI_i}) \\ &\quad \times \mathrm{sgn}(\mathrm{sgn}(W_{RI_i})X_{RI_i}(n) - \hat{\beta}_R(n)) \\ &+ e_I(n)\mathrm{sgn}(W_{RI_i})\mathrm{sgn}\left(\mathrm{sgn}(W_{RI_i})X_{IR_i^-}(n) - \hat{\beta}_I(n)\right)\Big]. \end{aligned}$$

The step-size parameter μ must be chosen properly to achieve the desired convergence rate. Notice that the two update terms in (14) resemble the real-imaginary coupled complex WM characteristics.

4. SIMULATIONS

To evaluate the filters adaptive line-enhancement [7] is used. The input of an eleven tap line enhancer is a complex exponential contaminated with α-stable noise [8] with dispersion $\gamma = 0.2$. The value of α runs from 1.3 to 2 (Gaussian noise). The noisy signal will be filtered using the marginal complex WM filter in (2), the marginal phase-coupled complex WM filter, the real-imaginary coupled complex WM filter, and a linear complex valued filter. To analyze the convergence properties of the algorithms, we plot the learning curves calculated as the average MSE of 1000 realizations of the experiment. Figure 1 shows the results for two values of α: (a) α=1.5, (b) α=2.0. The plot for the linear filter does not appear in 1(a) since it diverges.

Table 1. Average MSE using the LMS for Line enhancement. ($\mu = 0.005, \gamma = 0.2$)

Filter	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 2$
Noisy signal	45.2855	10.1473	3.3539	0.9326
Linear filter	∞	∞	∞	0.1152
Marginal complex WM	0.3759	0.3295	0.3332	0.3682
Real-imaginary coupled complex WM	0.2157	0.1601	0.1434	0.1438
Marginal phase coupled complex WM	0.1929	0.1316	0.1180	0.1154

For illustrative purposes the real part of the filter outputs are shown in Fig. 2. The plot shows 1000 samples of the filter output taken after the LMS algorithm has converged. The linear filter is not successful at filtering the impulsive noise while the complex WM filters are able to recover the original shape of the signal.

5. CONCLUSIONS

This paper introduced two efficient and well defined robust filtering structures and their LMS optimization.

The phase coupled complex WM uses phase coupling between weights and input samples. To calculate the output of this complex filter, a two-dimensional search in the complex plane is needed which results in a highly expensive implementation. To overcome this problem, the marginal

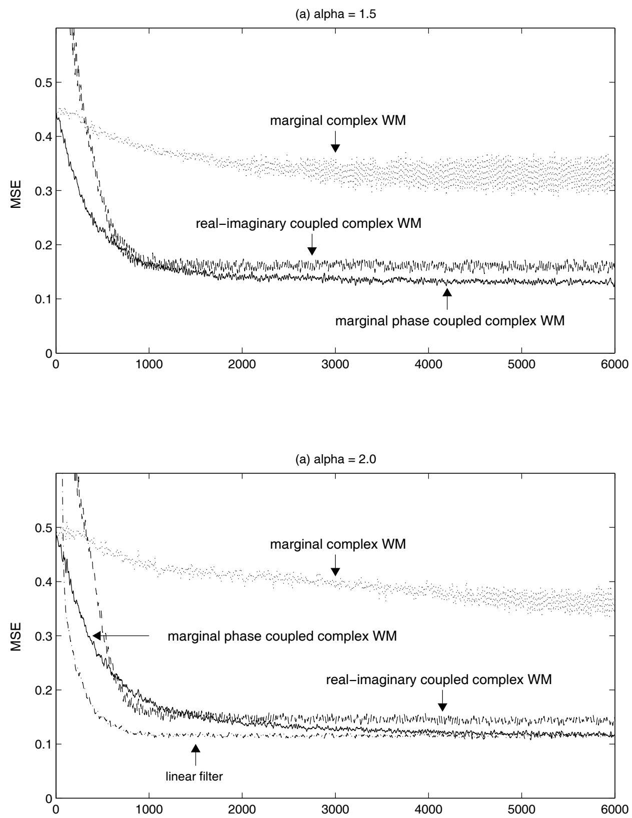

Fig. 1. Learning curves of the LMS algorithm for the complex WM filters and a linear filter for line enhancement in α-stable noise with dispersion $\gamma = 0.2$ and μ=0.005: (a) α=1.5, (b) α=2.

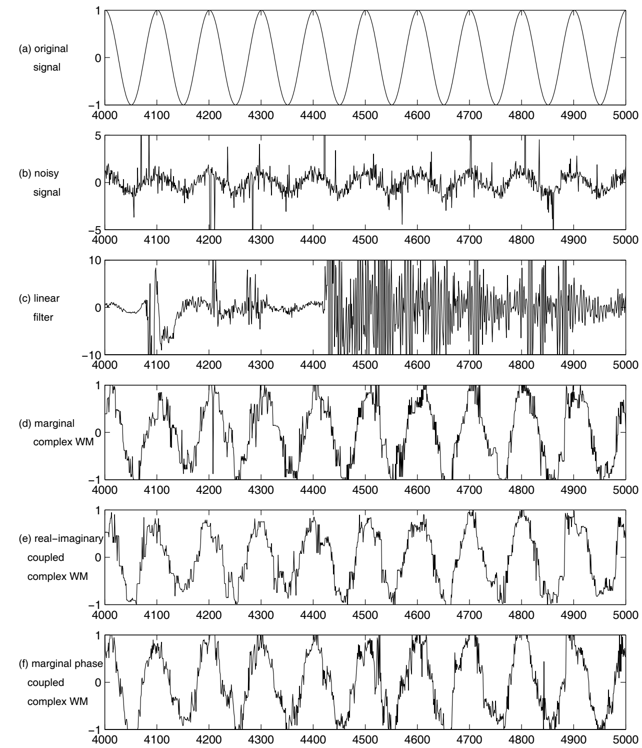

Fig. 2. Real part of the output of the filters for $\alpha = 1.7$, $\gamma = 0.2$ and $\mu = 0.005$.

phase coupled complex WM filter was proposed which separates the optimization problem found in the phase coupled complex WM into real and imaginary parts.

A second definition of complex-valued weighting in WM filters was also presented. It uses the natural coupling characteristic found in complex-valued linear filters. This definition is called real-imaginary coupled complex WM and exploits the correlation between real and imaginary domains.

The successful results obtained come from the fact that these new structures exploit the correlation information between the real and imaginary parts of the complex input samples instead of dealing with them independently. In general, these ideas not only can be used in complex-valued filtering framework based on median filters but in any other complex-valued structure based on other type of nonlinear filters.

6. REFERENCES

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