ON THE BENEFITS OF DELIBERATELY INTRODUCED BASEBAND NONLINEARITIES IN COMMUNICATION SYSTEMS

Hua Qian, Raviv Raich and G. Tong Zhou

School of Electrical and Computer Engineering
Georgia Institute of Technology, Atlanta, GA 30332-0250, USA

ABSTRACT

In this paper, we propose a baseband nonlinear transformation technique to improve the overall communication system performance, *under the peak power constraint.* A closed-form expression is derived for the signal-to-noise-and-distortion ratio (SNDR) of certain nonlinear transformations. A strategy for SNR-adaptive optimum clipping is proposed. For orthogonal frequency division multiplexing (OFDM), we show that the optimal clipping ratio leads to an SNDR improvement of 5-7 dB and accompanying decrease in symbol-error-rate. By applying an iterative symbol detection and clipping noise mitigation algorithm at the receiver, we demonstrate that clipping in OFDM can lead to large performance gains.

1. INTRODUCTION

Many components in a communication system have a peak power (or peak amplitude) constraint. For example, power amplifiers (PAs) are peak power limited. Denote by $x(n)$ a complex baseband information-bearing signal, and by σ_x^2 its variance (average power). Denote the received signal by

$$y(n) = g(x(n)) + v(n), \tag{1}$$

where $v(n)$ is zero-mean additive noise with variance σ_v^2, $g(\cdot)$ is a memoryless nonlinear transformation satisfying $|g(x)| \leq A_{\max}$, and $A_{\max}$ is the maximum amplitude that the transmitter can handle. For example, Fig. 1 (a) shows a soft limiter system where $x(n)$ is passed undistorted if $|x(n)| \leq A_{\max}$, but $g(x(n)) = A_{\max}\, e^{j\angle x(n)}$ if $|x(n)| > A_{\max}$. In most communication systems, $A_{\max}$ is governed by the PA; it is selected such that the signals are transmitted practically undistorted; i.e., the maximum signal amplitude must stay below $A_{\max}$ with a very high probability. Fig. 1 (b) shows a soft limiter with gain, where a linear gain $A_{\max}/A \geq 1$ exists. Fig. 1 (c) shows a nonlinear mapping that compresses the signal distribution and shifts it to the right. Fig. 1 (d) shows a hard limiter system, after which the amplitude information contained in $x(n)$ is completely discarded. In all four cases the maximum of $|g(x)|$ is $A_{\max}$.

We can write

$$g(x(n)) = \alpha x(n) + d(n), \tag{2}$$

where $d(n)$ is the distortion created by $g(\cdot)$, and α is chosen such that $d(n)$ is uncorrelated with $x(n)$.

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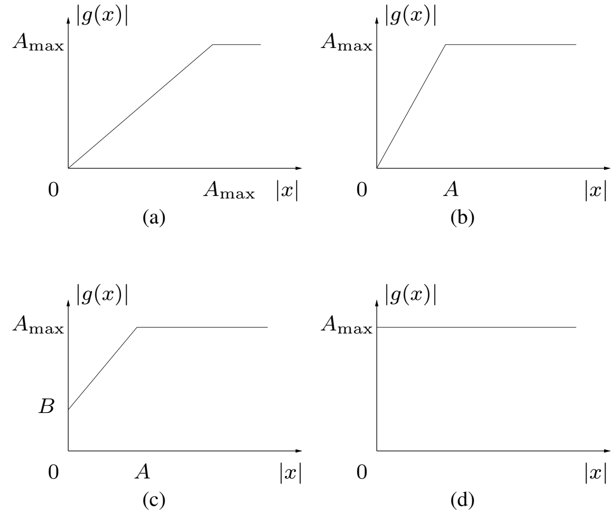

Fig. 1. Some nonlinear mappings under the peak amplitude constraint: (a) Soft limiter; (b) soft limiter with gain $A_{\max}/A \geq 1$; (c) limiter with amplitude translation and scaling; (d) hard limiter.

There are many $g(\cdot)$ functions that can ensure $|g(x(n))| \leq A_{\max}$. An interesting question to ask is, what constitutes a good nonlinear mapping? To answer this question, we use as performance metric, the signal-to-noise-and-distortion ratio (SNDR) [1, 2], defined as

$$\text{SNDR} = \frac{|\alpha|^2 \sigma_x^2}{\varepsilon_d + \sigma_v^2}, \tag{3}$$

where ε_d is the distortion power given by $E[|d(n)|^2]$.

Although nonlinearity is generally regarded as an impairment to a communication system, we argue in this paper that for a given SNR$= \sigma_x^2/\sigma_v^2$, a judicious choice of $g(\cdot)$ may lead to significant improvements in SNDR. We will show that for some low SNR levels, hard limiting may be more desirable than soft limiting, contrary to what one might think at first.

Our analysis has important implications for multi-carrier communication systems, where the peak-to-average power ratio (PAPR) of $x(n)$ is high, and PAPR reduction methods with distortion (e.g., soft limiter) are often of interest [3]. In the PAPR reduction literature, attention has been on peak power reduction methods. We argue that since the PA is peak power limited, and its efficiency is determined by the average output power $E[|g(x)|^2]$, it makes sense to investigate PAPR reduction methods that aim at increasing the average power while keeping the peak power fixed. SNDR offers a more complete picture than PAPR, by taking into account distortions that are generated by PAPR reduction methods.

Although symbol-error-rate (SER) is a more pertinent measure for communication system performance, for ease of theoretical analysis, we use SNDR improvements (relative to SNR) as the performance metric. We will show that with carefully introduced nonlinearity at the transmitter, together with iterative detection that mitigates the nonlinear distortions at the receiver, SER performance may be greatly improved.

2. SNDR WITH MEMORYLESS NONLINEAR MAPPING

Separating $g(x(n))$ into a linear term $\alpha x(n)$ and a distortion term $d(n)$ as in (2), we need

$$\alpha = \frac{E[x^*(n)g(x(n))]}{E[|x(n)|^2]} = \frac{E[x^*(n)g(x(n))]}{\sigma_x^2} \tag{4}$$

to ensure that $d(n)$ and $x(n)$ are uncorrelated [1,2]. The distortion power is given by

$$\varepsilon_d = E[|d(n)|^2] = E[|g(x(n)|^2] - |\alpha|^2 \sigma_x^2. \tag{5}$$

For a given distribution of $x(n)$ and a given $g(\cdot)$ function, both (4) and (5) can be evaluated.

Although many memoryless nonlinear transformations are of interest, we focus on the following nonlinear mapping:

$$g(x(n)) = \begin{cases} (B + G|x(n)|)\, e^{j\angle x(n)}, & |x(n)| \le A, \\ A_{\max}\, e^{j\angle x(n)}, & |x(n)| > A, \end{cases} \tag{6}$$

where $A = (A_{\max} - B)/G$. The soft limiter in Fig. 1(a) corresponds to (6) with $B = 0, G = 1$; the soft limiter with gain in Fig. 1(b) corresponds to (6) with $B = 0, G = A_{\max}/A > 1$; while the hard limiter in Fig. 1(c) is a special case of (6) with $B = A_{\max}$.

To visualize what the transformation in (6) does to the probability density function (PDF) of the signal amplitude, let us consider as an example, $r = |x|$ exponentially distributed with mean λ; i.e., the PDF $f(r) = \frac{1}{\lambda} e^{-\frac{r}{\lambda}}$, $r \ge 0$. Fig. 2(a) shows the PDF of $|x(n)|$, and Fig. 2(b) shows the PDF of $|g(x(n))|$ with $B = 0.5\lambda$, $A_{\max} = 3\lambda$ and $G = 1.25$. After the nonlinear mapping, the "center of gravity" of the signal amplitude is moved towards $A_{\max}$, implying that the PA is utilized more efficiently.

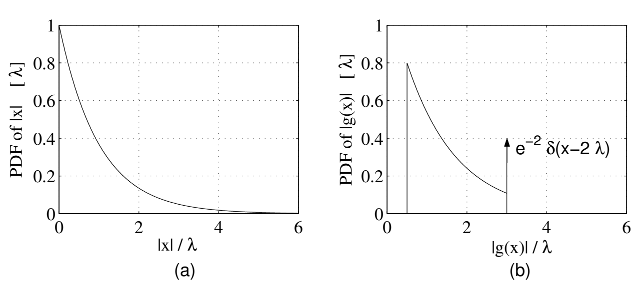

Fig. 2. PDF of $|x(n)|$ before and after the nonlinear mapping.

From now on, let us assume that $x(n)$ is complex Gaussian distributed with variance σ_x^2, which is of particular relevance to orthogonal frequency division multiplexing (OFDM). With a reasonably large number of sub-carriers, the time-domain OFDM signal $x(n)$ is approximately Gaussian distributed [4].

The complex Gaussian distributed signal is theoretically unlimited in amplitude although large amplitudes occur infrequently. In practice, we can set $A_{\max}$ to restrict the peak of the signal to be distorted with probability of at most ε; therefore, $A_{\max}$ is chosen such that $A_{\max}/\sigma_x$ is at a prescribed level.

Recall that $r = |x|$ is Rayleigh distributed with PDF

$$f(r) = \frac{2r}{\sigma_x^2}\, e^{-r^2/\sigma_x^2}, \qquad r \ge 0, \tag{7}$$

if x is complex Gaussian distributed. Assuming that $x(n)$ is independent, identically-distributed (i.i.d.), by solving

$$\Pr\left\{\max_{0 \le n \le N-1} |x(n)| > A_{\max}\right\} = 1 - \left(1 - e^{-\left(\frac{A_{\max}}{\sigma_x}\right)^2}\right)^N = \xi,$$

we infer that

$$A_{\max} = \sigma_x \sqrt{-\log(1 - \sqrt[N]{1-\xi})}. \tag{8}$$

Substituting (6) into (4) and using (7) to evaluate the expected value, we obtain

$$\alpha = \frac{A_{\max}}{\sigma_x}\frac{\sqrt{\pi}}{2} + G(1 - e^{-\gamma^2}) - G\gamma\left(\frac{\sqrt{\pi}}{2} - \sqrt{\pi}Q(\sqrt{2}\gamma)\right), \tag{9}$$

where γ is defined as the clipping ratio,

$$\gamma = \frac{A_{\max} - B}{G\sigma_x} = \frac{A}{\sigma_x}, \tag{10}$$

and $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}}\, dt$.

Based on (6) and (7), we infer that

$$\begin{aligned} E[|g(x(n))|^2] &= A_{\max}^2 + (B^2 - A_{\max}^2)\left(1 - e^{-\gamma^2}\right) \\ &\quad + 2BG\sigma_x\left(\frac{\sqrt{\pi}}{2} - \gamma e^{-\gamma^2} - \sqrt{\pi}Q(\sqrt{2}\gamma)\right) \\ &\quad + G^2\sigma_x^2\left(1 - (\gamma^2 + 1)e^{-\gamma^2}\right). \end{aligned} \tag{11}$$

Substituting (11) into (5), we can then find ε_d.

Next, we consider special cases of (6) corresponding to Fig. 1.

2.1. Soft limiter with gain

Setting $B = 0$, we have a soft limiter with gain

$$g(x(n)) = \begin{cases} Gx(n), & |x(n)| \le A, \\ A_{\max}\, e^{j\angle x(n)}, & |x(n)| > A, \end{cases} \tag{12}$$

see Fig. 1(b). With $\gamma = A_{\max}/(G\sigma_x)$, (9) simplifies to

$$\alpha = \frac{1}{\gamma}\frac{A_{\max}}{\sigma_x}\left(1 - e^{-\gamma^2} + \gamma\sqrt{\pi}\, Q(\sqrt{2}\gamma)\right), \tag{13}$$

and (11) reduces to

$$E[|g(x(n))|^2] = \left(\frac{A_{\max}}{\gamma}\right)^2 \left(1 - e^{-\gamma^2}\right). \tag{14}$$

Substituting (13) and (14) into (5) and (3), and simplifying, we obtain

$$\frac{\text{SNDR}}{\text{SNR}} = \frac{\left(\frac{A_{\max}}{\sigma_x}\right)^2 \varphi(\gamma)}{\left(\frac{A_{\max}}{\sigma_x}\right)^2 \text{SNR}\ \left(1 - e^{-\gamma^2} - \varphi(\gamma)\right) + \gamma^2}, \tag{15}$$

where SNR= σ_x^2/σ_v^2, and $\varphi(\gamma) = \left(1 - e^{-\gamma^2} + \sqrt{\pi}\gamma Q(\sqrt{2}\gamma)\right)^2$.

Converting both sides of (15) into dB scale (taking $10\log_{10}$), we plot in Fig. 3, improvements in SNDR; i.e., SNDR [dB] - SNR [dB], as a function of γ for prescribed levels of SNR. We see that for a given SNR, there is a range of γ values for which SNDR>SNR, implying that overall performance gain is possible with the nonlinear transformation (12), as compared to the case without any nonlinear distortion.

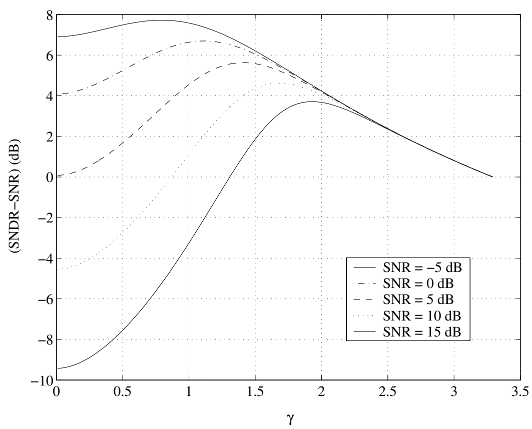

Fig. 3. Gain in SNDR as a function of γ for various SNRs.

We also see from Fig. 3 that for SNR between 0 and 10dB, maximum SNDR improvements (SNDR [dB] - SNR [dB]) are in the range 5-7 dB, by judiciously selecting γ. Define γ^* as the γ that maximizes the right hand side of (15) for a given SNR. Fig. 4 shows γ^* as a function of the SNR. We observe from Fig. 4 that when the SNR is low, it makes sense to clip more: although more distortion is generated, the average transmitted signal power is increased even more. From Fig. 4, we also observe that for very low SNRs, γ^* is close to 0, hinting on hard limiting.

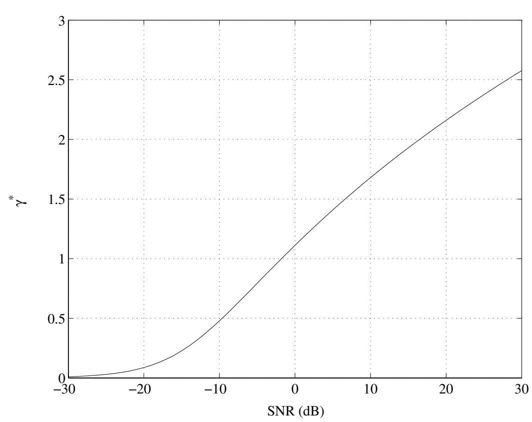

Fig. 4. Optimal γ as a function of SNR.

2.2. Hard limiter

When $B = A_{\max}$, we have the hard limiter case and the corresponding $\gamma = 0$; see Fig. 1(c). Eq. (9) simplifies to $\alpha = (A_{\max}/\sigma_x)\,(\sqrt{\pi}/2)$. Since $g(x(n))$ has a constant modulus, $E[|g(x(n))|^2] = A_{\max}^2$. As a result, $\varepsilon_d = (1-\pi/4)\sigma_x^2$, and the corresponding SNDR gain,

$$\frac{\text{SNDR}}{\text{SNR}} = \frac{\left(\frac{A_{\max}}{\sigma_x}\right)^2 \frac{\pi}{4}}{\left(\frac{A_{\max}}{\sigma_x}\right)^2 \text{SNR}\,(1-\frac{\pi}{4}) + 1}, \qquad (16)$$

which can also be obtained by taking the limit of (15) as $\gamma \to 0$.

Since $10\log_{10}(\pi/4) \approx -1$ dB, we infer from (16) that at a low SNR, the SNDR improvement in dB is approximately $20\log_{10}(A_{\max}/\sigma_x) - 1$ dB. For large $A_{\max}/\sigma_x$ values, this improvement can be significant. On the other hand, at a high SNR, the distortion introduced by the hard limiter becomes significant relative to the channel noise, leading to $\text{SNDR} < \text{SNR}$.

We would like to emphasize that unlike the PA nonlinearity, the nonlinearity that we apply is at the baseband and prior to pulse shaping, hence spectral regrowth (broadening) is not a problem.

3. APPLICATIONS TO OFDM

OFDM is well known for its robustness against frequency selective fading channels and for its high spectral efficiency. It has shown tremendous potential for high speed digital communication systems. It has been accepted as standards in many applications such as digital subscriber line [5] and digital audio/video broadcasting [6].

Denote by $\{X(k)\}_{k=0}^{N-1}$ the frequency domain OFDM signal drawn from a known constellation $\mathcal{C}$, and N is the number of subcarriers. Nyquist-rate sampled time domain OFDM signal is represented as

$$x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi kn}{N}}, \qquad 0 \le n \le N-1. \qquad (17)$$

It is well-known that $|x(n)|$ exhibits high peaks, especially for N large [3].

Suppose that the nonlinear transformation (12) is applied to (17) and we receive $y(n) = g(x(n)) + v(n) = \alpha x(n) + d(n) + v(n)$. In the frequency domain, we have $Y(k) = \alpha X(k) + D(k) + V(k)$. From (13), α is known once $A_{\max}/\sigma_x$ and γ are available. Since $X(k)$ belongs to a known constellation $\mathcal{C}$, and the distortion mechanism $g(\cdot)$ is known, both the symbols $X(k)$ and the distortion term $d(n)$ can be iteratively estimated. Similar to [7], we adopt the following iterative symbol detection algorithm:

Initialize with $q = 0$, $D^{(0)}(k) = 0$, for $k = 0, \ldots, N-1$.

$$X^{(q+1)}(k) = \arg\min_{X(k)\in\mathcal{C}} |Y(k) - D^{(q)}(k) - \alpha\, X(k)|^2, \quad (18)$$

$$x^{(q+1)}(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X^{(q+1)}(k)\, e^{j2\pi nk/N}, \qquad (19)$$

$$d^{(q+1)}(n) = y(n) - \alpha x^{(q+1)}(n), \qquad (20)$$

$$D^{(q+1)}(k) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} d^{(q+1)}(n)\, e^{-j2\pi nk/N}. \qquad (21)$$

The difference between the above algorithm and [7] is that in the above algorithm, α is taken in to consideration, whereas in [7], α is set to 1, which is appropriate only when mild clipping is applied.

4. SIMULATIONS

For both examples, we generated the QPSK sequence $\{X(k)\}_{k=0}^{N-1}$ with $N = 512$ and $X(k) \in \mathcal{C} = \{\sigma_x e^{\pm j\frac{\pi}{4}}, \sigma_x e^{\pm j\frac{3\pi}{4}}\}$. $A_{\max}$ of the system is obtained from (8) with $\varepsilon = 0.01$. We obtained $x(n)$ according to (17). Afterwards, the soft limiter with gain in (12) is applied. White complex Gaussian noise $v(n)$ is then added to $g(x(n))$ to produce $y(n)$. What we refer to as unclipped OFDM is actually $g(x(n))$ in (12) with $G = 1$ (or $A = A_{\max}$).

Example 1. The benefit of clipping with an SNR-adaptive γ^*.

If we treat the distortion as a part of the noise, we can apply a simple QPSK decoder; i.e., (18) with $q = 0$ (no iterations):

$$X^{(1)}(k) = \arg\min_{X(k)\in\mathcal{C}} |Y(k) - \alpha\, X(k)|^2. \qquad (22)$$

We compare $X^{(1)}(k)$ with $X(k)$ to calculate the SER.

Fig. 5 shows the SER for unclipped QPSK-OFDM, clipping with fixed $\gamma = 0.5,\ 0.7,\ \ldots,\ 1.7$, and clipping with the optimal γ^* as shown in Fig. 4. We see from Fig. 5 that SNR-adaptive clipping consistently outperforms the unclipped one; the SNR gain was 5.5 dB at the SER $= 10^{-4}$ level. With a fixed clipping ratio γ, SER improvements were obtained at low SNR levels, but SER deteriorations occurred at high SNR. Note that no additional complexity, as compared to the unclipped case, is required for the decoder in (22).

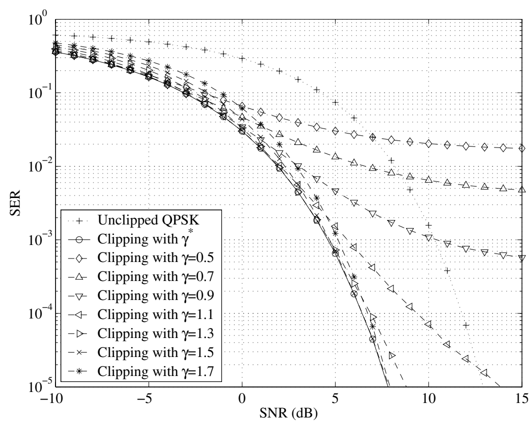

Fig. 5. SER vs. SNR curves for unclipped OFDM, clipped OFDM with $\gamma = 0.5,\ 0.7,\ \ldots,\ 1.7$, and clipped OFDM with $\gamma = \gamma^*$.

Example 2. The benefit of clipping noise mitigation.

For Fig. 6, we implemented the iterative procedure (18)-(21), for soft clipping (12) with a fixed $\gamma = 0.6$. SER is calculated by comparing $X^{(q+1)}(k)$ with $X(k)$ for $q = 0,\ 1,\ 2,\ 3,$ and 10. For comparison purposes, we also show SER of the unclipped OFDM signal, and that corresponding to clipped OFDM with the optimal γ^* but without iterations (as in Example 1). From Fig. 6, we see that even with a fixed γ, SER can be significantly improved from the unclipped case, if iterative clipping noise mitigation is performed at the receiver. At the SER $= 10^{-4}$ level, an SNR gain of 8.5 dB was achieved. The result is even better than using the optimal γ^* but without iterations. We also observe that at low SNRs, iterations offer little improvement since the starting estimate $X^{(1)}(k)$ is often poor.

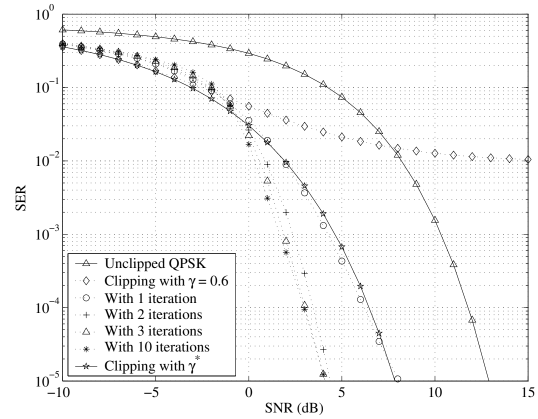

Fig. 6. SER for unclipped OFDM; clipped OFDM with $\gamma = 0.6$ for iterations 0, 1, 2, and 10; and clipped OFDM with $\gamma = \gamma^*$.

5. CONCLUSIONS

In this paper, we proposed a baseband nonlinear transformation technique to improve the overall communication system performance, *under the peak power constraint.* A closed-form expression is derived for the SNDR of certain nonlinear transformations. A strategy for SNR-adaptive optimum clipping is proposed. For OFDM with soft clipping, we showed that the optimal clipping ratio leads to an SNDR improvement of 5-7 dB and accompanying decrease in SER. By applying an iterative symbol detection and clipping noise mitigation algorithm at the receiver, we demonstrated that clipping in OFDM can be very beneficial.

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