FUZZY LUM FILTERS

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ABSTRACT

Fuzzy ranks are a component of *fuzzy ordering* theory developed to incorporate sample spread information into sample ranking. In this paper, the well-known *lower-upper-middle* (LUM) filters are generalized by utilizing the *fuzzy* ranks, which is called *fuzzy* LUM (F-LUM) filters. We develop the statistical and deterministic properties of the F-LUM filters and present their performance in image denoising and enhancing applications. As shown by the experimental results, the F-LUM filters inherit the simplicity and versatility of the LUM filters such that they can perform signal smoothing, sharpening and outlier rejection using a single filter structure, while having great improvement in detail preservation and robustness to noise.

1. INTRODUCTION

The *fuzzy ordering* theory was developed to jointly utilize the samples' spatial, rank-order and spread (or diversity) information in signal filtering [1, 2]. The theory introduces a real-valued (instead of binary) *spatial-rank* (SR) matrix to describe the relationship between each spatial sample and order statistic. This matrix is called *fuzzy* SR matrix, based on which the concepts of *fuzzy samples*, *fuzzy order statistics* and *fuzzy ranks* are developed. In our previous work [2, 3, 4, 5, 6], fuzzy samples and fuzzy order statistics are exploited to generalize a series of conventional filters, such as identity, median, weighted median and *rank condition rank selection* (RCRS) filters. These generalized filters have demonstrated good performance in a wide range of applications such as image denoising, zooming, sharpening and deblocking.

Similar to the fuzzy samples and fuzzy order statistics, fuzzy ranks also possess interesting properties such that they represent not only the rank-order information but also the spread information. Nevertheless, fuzzy ranks haven't been fully investigated or utilized. Therefore, we are motivated to exploit them in generalizing other rank-order-based filters.

A well-known class of rank-order-based filters are the *lower-upper-middle* (LUM) filters [7], whose output is defined by

$$y = \begin{cases} x_{(k)}, & x_c < x_{(k)}, \\ x_{(l)}, & x_{(l)} < x_c \le t_l, \\ x_{(N-l+1)}, & t_l < x_c < x_{(N-l+1)}, \\ x_{(N-k+1)}, & x_c > x_{(N-k+1)}, \\ x_c, & \text{otherwise}, \end{cases} \tag{1}$$

where x_c is the the center sample in the filtering window of size N, $k \le l \le (N+1)/2$, and $t_l = (x_{(l)} + x_{(N-l+1)})/2$. The LUM filters can perform purely smoothing, by setting $l = (N+1)/2$, or purely sharpening, by setting $k = 1$, or a joint function for other settings. The smoothing and sharpening level can be controlled by adjusting the parameters k and l, respectively. However, since the LUM filters are based on the samples' crisp rank-order information and the choice of filter parameters are confined by the filtering window size, their flexibility in detail preservation and robustness to noise are limited. In this paper, we show that by simply replacing the crisp ranks with the fuzzy ranks, the performance of the LUM filters can be greatly improved in both aspects without sacrificing their simplicity or versatility.

2. FUZZY ORDERING THEORY AND FUZZY RANKS

In the fuzzy ordering theory, we consider a spatial ordered sample vector $\mathbf{x}_\ell = [x_1, x_2, \ldots, x_N]$ and its corresponding rank ordered vector $\mathbf{x}_L = [x_{(1)}, x_{(2)}, \ldots, x_{(N)}]$, where $x_{(1)} \le x_{(2)} \le \cdots \le x_{(N)}$ are the order statistics. The fuzzy SR matrix is defined by

$$\tilde{\mathbf{R}} = \begin{bmatrix} \tilde{R}_{1,(1)} & \ldots & \tilde{R}_{1,(N)} \\ \vdots & \ddots & \vdots \\ \tilde{R}_{N,(1)} & \ldots & \tilde{R}_{N,(N)} \end{bmatrix}, \tag{2}$$

where $\tilde{R}_{i,(j)} = \mu_{\tilde{R}}(x_i, x_{(j)}) \in [0,1]$, $i, j = 1, 2, \ldots, N$ and $\mu_{\tilde{R}}(a,b)$ is a real-valued membership function that describes the relationship between the inputs a and b with the following restrains:

1. $\lim_{|a-b|\to 0} \mu_{\tilde{R}}(a,b) = 1$,
2. $\lim_{|a-b|\to \infty} \mu_{\tilde{R}}(a,b) = 0$,
3. $\mu_{\tilde{R}}(a_1,b_1) \ge \mu_{\tilde{R}}(a_2,b_2)$, if $|a_1 - b_1| \le |a_2 - b_2|$.

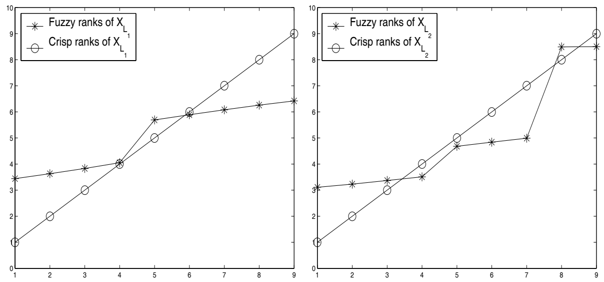
The fuzzy ranks are defined as follows. Let $\mathbf{r} = [r_1, r_2, \ldots, r_N]$ be the rank vector of the crisp samples, where r_i is the rank of x_i. Then the fuzzy rank vector $\tilde{\mathbf{r}} = [\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_N]$ is defined by $\tilde{\mathbf{r}}^T = \tilde{\mathbf{R}}^\ell \mathbf{I}^T$, where $\tilde{\mathbf{R}}^\ell$ is the row normalized fuzzy SR matrix, and $\mathbf{I} = [1, 2 \ldots N]$ is the crisp spatial index vector. Thus, the expression for the fuzzy rank $\tilde{r}_i$ is:

$$\tilde{r}_i = \frac{\sum_{k=1}^N k\tilde{R}_{i,(k)}}{\sum_{k=1}^N \tilde{R}_{i,(k)}}. \tag{3}$$

The following theorem is proved in our recent work [6]:

Order Invariant Theorem: The fuzzy ranks obey the same order as the crisp ranks, i.e., $\tilde{r}_i < \tilde{r}_j$ if and only if $r_i < r_j$, given that the membership function $\mu_{\tilde{R}}(\cdot,\cdot)$ is such that $C(x,t,\Delta t) = \frac{\mu_{\tilde{R}}(x,t+\Delta t)}{\mu_{\tilde{R}}(x,t)}$ is a monotonically non-decreasing function of x, for $\forall\, t \in R$ and $\Delta t \in R^+$.

This theorem implies that the fuzzy ranks and the crisp ranks represent consistent rank order. Hence, generalizing the rank-order-based filters by using fuzzy ranks are theoretically justified. In addition, fuzzy ranks carry the spread information of the crisp samples, as shown in the following example. Consider two sample



(a) (b)

Fig. 1. Spread information represented by the fuzzy ranks for $\mathbf{x}_{\ell_1}$ and $\mathbf{x}_{\ell_2}$. (a) $\mathbf{r}_1$ and $\tilde{\mathbf{r}}_1$. (b) $\mathbf{r}_2$ and $\tilde{\mathbf{r}}_2$.

vectors $\mathbf{x}_{\ell_1}$ = [1.1, 1.2, 1.3, 1.4, 2.1, 2.2, 2.3, 2.4, 2.5], where there is no sample outliers, and $\mathbf{x}_{\ell_2}$ = [1.1, 1.2, 1.3, 1.4, 2.1, 2.2, 2.3, 12.4, 12.5], where there are two sample outliers. The crisp rank vectors of both $\mathbf{x}_{\ell_1}$ and $\mathbf{x}_{\ell_2}$ are $\mathbf{r_1} = \mathbf{r_2} = [1, 2, \ldots, 9]$, while the fuzzy rank vectors are $\tilde{\mathbf{r}}_\mathbf{1}$=[3.44, 3.63, 3.83, 4.05, 5.69, 5.89, 6.08, 6.26, 6.42] and $\tilde{\mathbf{r}}_\mathbf{2}$ = [3.11, 3.23, 3.37, 3.51, 4.68, 4.84, 4.99, 8.49, 8.50] [1]. As shown in Fig. 1, the crisp ranks are independent of the sample diversity, while the fuzzy ranks reflect both the samples' rank order and diversity. It is also important to note that similar samples have similar fuzzy ranks of value approximately equal to the median of their crisp ranks. For example, in $\tilde{\mathbf{r}}_\mathbf{1}$, all the fuzzy ranks are around the global median "5". In $\tilde{\mathbf{r}}_\mathbf{2}$, the first seven fuzzy ranks are around their local median "4", while the fuzzy ranks of the two outliers are around their local median "8.5" (note they are also very close to the crisp ranks "8" and "9"). These properties introduce a number of desirable features that can be exploited in filter generalizations.

3. DEFINITION OF F-LUM FILTERS

The output of the F-LUM filters is defined as follows,

$$y^* = \begin{cases} x_{(k)}, & \tilde{r}_c < k, \\ x_{(l)}, & l < \tilde{r}_c < h, \\ x_{(N-l+1)}, & N-h+1 < \tilde{r}_c < N-l+1, \\ x_{(N-k+1)}, & \tilde{r}_c > N-k+1, \\ x_c, & \text{otherwise}, \end{cases} \quad (4)$$

where $k \leq l \leq h \leq (N+1)/2$ and $\tilde{r}_c$ is the fuzzy rank of the center sample x_c.

Similarly to the LUM filter, setting $l = h = (N+1)/2$ produces a F-LUM filter that performs only smoothing and is thus called F-LUM smoother. It does not modify x_c when $\tilde{r}_c$ falls into the range $[k, N-k+1]$; otherwise, it replaces x_c by $x_{(k)}$ or $x_{(N-k+1)}$ according to which one is closer to x_c. The F-LUM sharpener is defined by setting $k = 1$. Unlike the LUM sharpener, the F-LUM sharpener has two parameters l and h, which define three regions, i.e., $[1, l]$, $[N-l+1, N]$ and $[h, N-h+1]$. x_c will not be modified when $\tilde{r}_c$ is in one of the three regions; otherwise, it is replaced by $x_{(l)}$ or $x_{(N-l+1)}$ accordingly. For other settings, the F-LUM filters can perform joint smoothing and sharpening. The parameter k controls the smoothing level, l controls the sharpening level, while the new parameter h is introduced to avoid enhancing small variations presented in the signal, the advantages of which will be shown in the following sections.

[1]These fuzzy ranks are computed by using Gaussian membership function which is defined by $\mu_G(a,b) = \exp[-(a-b)^2/(2\sigma^2)]$, the spread parameter $\sigma = 1$. Gaussian membership function satisfies the order invariant condition.

4. PROPERTIES OF F-LUM FILTERS

The F-LUM filters have a number of desirable deterministic and statistic properties. In this paper, due to the space constrains, we only present the impulsive noise *breakdown* and *false-alarm* probabilities of the F-LUM smoothers to show their advantages in impulse removal and detail preservation. We also analyze the *small variation preservation* property of the F-LUM sharpeners to show how they prevent the distortion of fine details and false enhancement of noise.

4.1. Breakdown Probability of F-LUM Smoother

The breakdown probability is the probability of outputting an impulse given a certain probability of impulse appearing in the input. It is an indication of the impulse removal capability of a filter. From the previous example, we can intuitively see that when x_c is an outlier, $\tilde{r}_c$ is approximately equal to the crisp rank and tends to be outside of the range $[k, N-k+1]$. Thus the F-LUM smoothers have behavior similar to the LUM smoothers. Therefore, they should have similar breakdown probabilities for the same k.

To prove this, let y be the output of the LUM smoother. When the positive and negative impulses are equally likely to appear in the input, the breakdown probability of the LUM smoother is [7]:

$$\begin{aligned} &Pr(y = \pm\infty) = 2 \cdot Pr(y = -\infty) \\ &= 2 \cdot Pr(x_c = -\infty)Pr(x_{(k)} = -\infty | x_c = -\infty) \\ &\quad + Pr(x_c \neq -\infty)Pr(x_{(N-k+1)} = -\infty | x_c \neq -\infty). \end{aligned}$$

To analyze the breakdown probability of the F-LUM smoother, we assume $\mu_{\tilde{R}}(-\infty, -\infty) = \mu_{\tilde{R}}(+\infty, +\infty) = 1$, $\mu_{\tilde{R}}(-\infty, +\infty) = \mu_{\tilde{R}}(\pm\infty, x) = 0$, where $x \neq \pm\infty$. Let y^* be the output of the F-LUM smoother. Since $Pr(y^* = \pm\infty) = 2 \cdot Pr(y^* = -\infty)$, we need only consider $Pr(y^* = -\infty)$ in the following two cases, where we suppose there are L samples equal to $-\infty$:
(I) $x_c = -\infty$, $\tilde{r}_c = \frac{L+1}{2}$, where $L \in [1, N]$, then

$$\begin{aligned} &Pr(y^* = -\infty | x_c = -\infty) \\ &= Pr(\tilde{r}_c \leq k, x_{(k)} = -\infty | x_c = -\infty) \\ &\quad + Pr(k < \tilde{r}_c < N-k+1 | x_c = -\infty) \\ &\quad + Pr(\tilde{r}_c \geq N-k+1, x_{(N-k+1)} = -\infty | x_c = -\infty) \\ &= Pr(k \leq L \leq 2k-1 | x_c = -\infty) \\ &\quad + Pr(2k-1 < L \leq N | x_c = -\infty) + 0 \\ &= Pr(x_{(k)} = -\infty | x_c = -\infty). \end{aligned}$$

(II) $x_c \neq -\infty$, $\tilde{r}_c > L$, where $L \in [0, N-1]$, then

$$\begin{aligned} &Pr(y^* = -\infty) | x_c \neq -\infty) \\ &= Pr(\tilde{r}_c \leq k, x_{(k)} = -\infty | x_c \neq -\infty) \\ &\quad + Pr(\tilde{r}_c \geq N-k+1, x_{(N-k+1)} = -\infty | x_c \neq -\infty) \\ &= Pr(\tilde{r}_c < k \leq L | \tilde{r}_c > L) + Pr(\tilde{r}_c \geq L \geq N-k+1 | \tilde{r}_c > L) \\ &= 0 + Pr(L \geq N-k+1 | \tilde{r}_c > L) \\ &= Pr(x_{(N-k+1)} = -\infty | x_c \neq -\infty). \end{aligned}$$

From (I) (II) we have $Pr(y^* = -\infty) = Pr(y = -\infty)$, so $Pr(y^* = \pm\infty) = Pr(y = \pm\infty)$, which means F-LUM and LUM smoother have the same break-down probability. ■

4.2. False-alarm Probability of F-LUM Smoother

The false-alarm probability is the probability of an uncorrupted sample being modified by the filter. It is an indication of the detail preservation capability of a filter. As we can see from the previous example, $\tilde{r}_c$ tends to be near the median when x_c is not an outlier, even though the crisp rank of x_c may fall outside the range $[k, N-k+1]$. Thus, the F-LUM smoothers can preserve the original details better by avoiding the modification in the absence of outliers. Hence, the F-LUM smoothers generally have lower false-alarm probability than the LUM smoothers. For more insights, assume the input samples are i.i.d. In the case where i samples equal $-\infty$ and j samples equal $+\infty$, the crisp rank of the center sample r_c is uniformly distributed in $[i+1, N-j]$. So the false-alarm probability of the LUM smoother is

$$1 - \sum_{i=0}^{N-1} \sum_{j=0}^{N-1-i} P(i,j)Q(i,j),$$

where $Q(i,j) = \frac{max[0, min(N-k+1, N-j) - max(k, i+1)]}{N-i-j}$ is the conditional probability of r_c being within $[k, N-k+1]$, $P(i,j)$ is the probability that i samples equal $-\infty$ and j samples equal $+\infty$. Similarly, the false-alarm probability of the F-LUM smoother is

$$1 - \sum_{i=0}^{N-1} \sum_{j=0}^{N-1-i} P(i,j)\tilde{Q}(i,j),$$

where $\tilde{Q}(i,j)$ is the conditional probability that $\tilde{r}_c$ is within $[k, N-k+1]$. Note under each condition, $\tilde{r}_c = \frac{\sum_{k=i+1}^{N-j} k\tilde{R}_{c,(k)}}{\sum_{k=i+1}^{N-j} \tilde{R}_{c,(k)}}$ can be approximated by $\frac{\sum_{k=i+1}^{N-j} k\tilde{R}_{c,k}}{\sum_{k=i+1}^{N-j} \tilde{R}_{c,k}}$ in distribution. Moreover, the $\alpha_k = \tilde{R}_{c,k}$ terms are identically distributed and $\alpha_k \approx 1$ when the spread parameter of the membership function is sufficiently large. Thus we have

$$\tilde{r}_c \approx \frac{\sum_{k=i+1}^{N-j} k\alpha_k}{N-j-i} = \frac{N-j+i+1}{2} E[\alpha_k] + \frac{s_{i,j}}{N-j-i} \cdot \frac{\sum_{k=i+1}^{N-j} Y_k}{s_{i,j}},$$

where $Y_k = k(\alpha_k - E[\alpha_k])$, $s_{i,j}^2 = \sum_k E[Y_k^2]$. By the *Lindeberg-Feller Central Limit Theorem* [8], $W_{i,j} = \frac{\sum_{k=i+1}^{N-j} Y_k}{s_{i,j}}$ tends to follow the standard normal distribution $N(0,1)$. Since $E[\alpha_k] \approx 1$, the distribution of $\tilde{r}_c$ can be approximated by $N(\frac{N-j+i+1}{2}, \frac{s_{i,j}}{N-j-i})$. If the input samples are Gaussian distributed with variance ξ^2 and the Gaussian membership function with spread parameter σ is used, it is not difficult to obtain $Var[\alpha_k] = \frac{\sigma}{\sqrt{4\xi^2+\sigma^2}} - (\frac{\sigma}{\sqrt{2\xi^2+\sigma^2}})^2$, and express $s_{i,j}$ in terms of $Var[\alpha_k]$. Then, we can compute $\tilde{Q}(i,j)$ and obtain the approximated false-alarm probability of the F-LUM smoother. The theoretical results of both LUM and F-LUM smoothers for $\xi = 10, \sigma = 25$ and 2% impulsive noise are shown in Fig. 3, which is consistent with our intuitive analysis.

4.3. Small Variation Preservation of the F-LUM Sharpener

Small variations of the image data often result from the fine details of the image or from the low-level noise in a flat region. They usually manifest themselves as pulses with gentle slopes. Regardless of the heights, these gentle slopes tend to be converted into ideal steps by the LUM sharpeners [7]. Thus, the LUM sharpener may severely alter the original structure of the image by distorting the fine details or introducing noisy patches in flat regions.

By introducing the new parameter h, F-LUM sharpeners preserve the input when $\tilde{r}_c \in [h, N-h+1]$. Note that $\tilde{r}_c$ is close

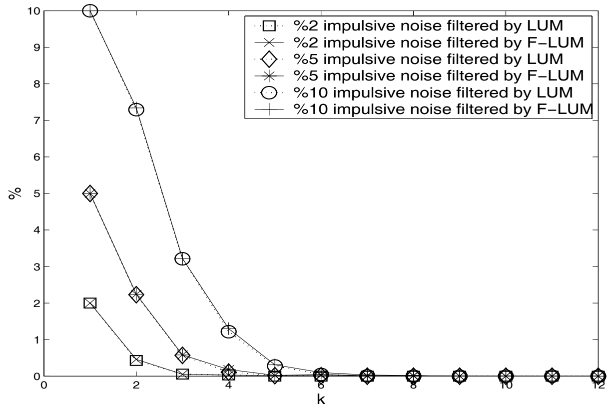


Fig. 2. Breakdown probability of LUM and F-LUM smoothers applied on image House.

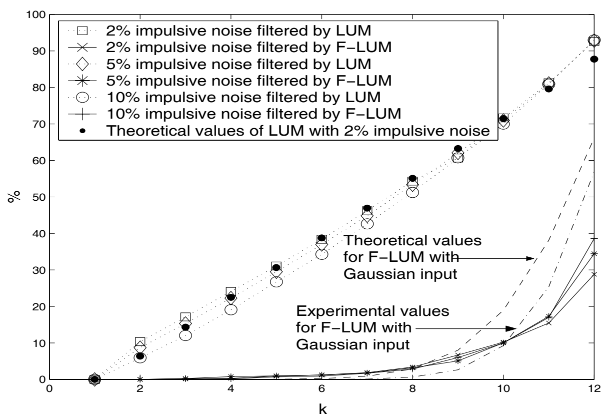


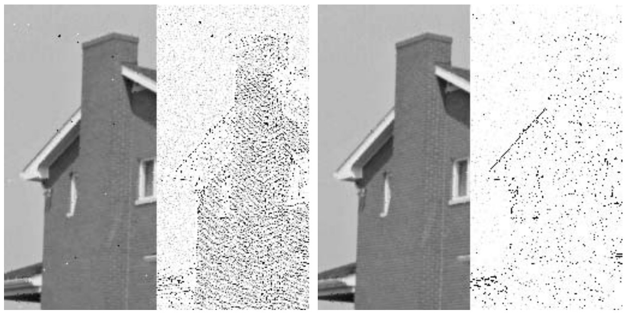
Fig. 3. False-alarm probability of LUM and F-LUM smoothers applied on image House and Gaussian input with distribution $N(128, 10)$.

to the median, and thus very likely to be within $[h, N-h+1]$, as long as the samples' diversity is small. Therefore, the F-LUM sharpeners can preserve the small variations presented in the signal and avoid the undesired enhancement mentioned above, while still perform desirable enhancing along the key edges. Using the crisp rank in the same framework, however, cannot achieve the same goal since the crisp ranks are independent of the sample diversity.

5. EXPERIMENTAL RESULTS

To evaluate the performance of the LUM and F-LUM smoothers (sharpeners), we perform experiments on real images. The test images' sizes are 256×256, the filtering window size is 5×5 and a Gaussian membership function is used. To compare the performance of LUM and F-LUM smoothers, we apply them on the image House corrupted by different levels of impulsive noise. For each value of k, the spread parameter σ is optimized [2] to yield the minimal mean absolute error (MAE). The experimental breakdown and false-alarm probabilities of each filter are plotted in Fig. 2 and 3, respectively. As can be seen, for the same

[2]The parameter σ can be optimized using a simple stochastic gradient approach on a training image such as Lenna, due to the space constrains, we do not present the details in this paper.



(a) (b)

Fig. 4. Output and error image of the (a) LUM and (b) F-LUM smoother applied on House image corrupted by 5% impulsive noise, where $\sigma = 70$, k is optimized for both filters to yield minimal MAE. The F-LUM smoother performs better in both noise removal and detail preservation.

k, F-LUM smoother has the same breakdown probability as the LUM smoother, while achieving much lower false-alarm probability. The experimental false-alarm probabilities of the F-LUM smoothers ($\sigma = 25$) applied on Gaussian input ($\xi = 10$) corrupted by 2% impulsive noise are also shown in Fig. 3, where the theoretical results approximate the experimental data well especially for k small. Thus, by choosing appropriate k and σ, F-LUM smoother can excel in both noise removal and detail preservation. Figure 4 shows such an example.

To compare the performance of the LUM and F-LUM sharpener, we test the Mandrill and the Lenna images (Lenna is corrupted by zero-mean Gaussian noise with variance 25). The results are shown in Fig. 5. We can see that the LUM sharpener distorts the fine textures of the Mandrill's fur around the face and below the nose; while the F-LUM sharpener enhances the image contrast and preserves the details well. For the Lenna image, the LUM sharpener introduces many noisy patches on Lenna's face; while F-LUM sharpener preserves the smoothness of the face and enhances the edges of the hat and the feathers.

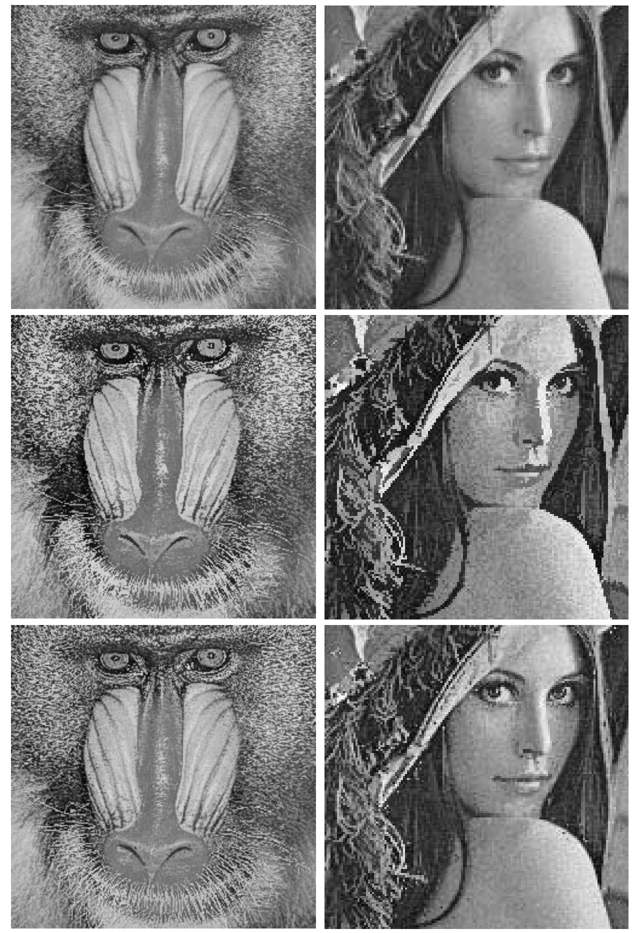


Fig. 5. Output of the LUM and F-LUM sharpener on Mandrill and Lenna images. Row1: input images, Row2: output of the LUM sharpener, Row3: output of the F-LUM sharpener. $k = l = 1, h = 9, \sigma = 20$.

6. CONCLUSION

The F-LUM filters, by utilizing the fuzzy ranks, achieve better detail (or smoothness) preservation than standard LUM filters in both smoothing and sharpening process without sacrificing their noise removal and contrast enhancement capability. The spread parameter of the membership function and the new filter parameter h provide more flexibility in filter design while retaining the filter's simplicity. Our current research include evaluating the the general F-LUM filters in joint smoothing and sharpening process. We believe they should have good performance in all the aspects of outlier rejection, edge enhancement and structure preservation.

7. REFERENCES

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