INSTANTANEOUS FREQUENCY ESTIMATION USING DOUBLE-SIDED EXPONENTIALLY FORGETTING TRANSFORM

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Abstract

In this paper, we propose a new type of time-frequency distribution named double-sided exponentially forgetting transform (DSEFT). And the algorithm of instantaneous frequency estimation using DSEFT is presented. It is shown that the DSEFT offers good time-frequency resolution and that stable reconstruction of signal from samples of the DSEFT is possible. Compared with other forms of time-frequency representation, DSEFT is highly computational efficient, since iterative computation can be used. Analysis shows that DSEFT is superior to exponential forgetting transform as an instantaneous frequency estimator. The bias and the mean square deviation of the estimator are discussed and its validness is verified by the simulations.

1. Introduction

Although there are many ways to estimate the instantaneous frequency(IF) of a non-stationary signal, generally they can be divided into three approaches. The first assumes the signal to be locally stationary within a sliding window and the standard frequency estimator is used. The second approach is based on the assumption of linear variation of the signal IF, the Kay-Tretter IF estimator [1] belongs to this type. And the third approach makes no particular assumption and the frequency estimators are derived from the general time-frequency representation [2,3,4,5].

Since the computation of time-frequency distribution is a time consuming task, it is natural that the idea to perform the calculation iteratively. Nowadays most of the recursive algorithm of time-frequency distribution are based on all pole moving window which progressively incorporation the new samples and produce the "fading memory" effect. Being a special case of this type, exponentially forgetting transform (EFT) [6,7] has been studied. Although computationally efficient, EFT is not practically useful for IF estimation, since unpredictable excursion of IF representation will result, and the smaller the forgetting factor the greater the excursion will be, a fact that can be seen in the simulations.

The concept of a new type of time-frequency representation called double-sided exponentially forgetting transform (DSETF) is presented in this paper. The symmetrical exponential window will introduce no excursion in IF representation. And computational efficiency is achieved by the use of recursive computation. In the later section the algorithm of IF estimation using DSEFT is explored. Finally its performance is discussed.

2 Double-Sided Exponentially Forgetting Transform

2.1 Definition

With the double-sided exponential window: $h(n)=e^{-\lambda|n|}$, where the λ is the forgetting factor, the discrete time DSEFT becomes

$$Y_{n,\omega}=\sum_{m=-\infty}^{\infty}x(m)e^{-\lambda|m-n|}e^{-j\omega m} \tag{1}$$

which is essentially the Fourier transform of the exponentially weighted (forgotten) signal history. Due to this 'forgetting' property, we shall, for convenience, use the name double-sided exponentially forgetting transform.

And the for DSEFT, $Y_{n,\omega}$ can be divided into two parts:

$$Y_{n,\omega}=Y^1{}_{n,\omega}+Y^2{}_{n,\omega}$$

where

$$Y^1{}_{n,\omega} = \sum_{m=-\infty}^{n} x(m) e^{\lambda(m-n)} e^{-j\omega m} \tag{2}$$

$$Y^2{}_{n,\omega} = \sum_{m=n+1}^{\infty} x(m) e^{\lambda(n-m)} e^{-j\omega m} \tag{3}$$

The use of double-sided exponential window makes it possible the recursive computation:

$$\begin{aligned} Y_{n+1,\omega} &= Y^1{}_{n+1,\omega} + Y^2{}_{n+1,\omega} = x(n+1)\cdot e^{-j\omega(n+1)} + e^{-\lambda} Y^1{}_{n,\omega} \\ &+ e^{\lambda}\cdot Y^2{}_{n,\omega} - x(n+1)\cdot e^{-j\omega(n+1)} \\ &= e^{-\lambda} Y^1{}_{n,\omega} + e^{\lambda}\cdot Y^2{}_{n,\omega} \end{aligned} \tag{4}$$

2.2 Signal Reconstruction

The signal can be recovered from its DSEFT. Suppose that DSEFT is known at ω_0, the signal $x(t)$ can be uniquely determined,

$$\begin{aligned} x(t) = \frac{1}{2}\cdot e^{j\omega_0 t} \Big\{ &\frac{\partial}{\partial t}\left[Y^1(n,\omega_0;\beta) - Y^2(n,\omega_0;\beta) \right] \\ &+ \beta \cdot \left[Y^1(n,\omega_0;\beta) + Y^2(n,\omega_0;\beta) \right] \Big\} \end{aligned} \tag{5}$$

3. Performance of IF Estimate Using DSEFT

Consider the problem of IF estimation from the discrete time observations,

$$x(n) = s(n) + \varepsilon(n)$$

With $s(n) = Ae^{j\phi(n)}$, where A is the magnitude and $\phi(n)$ the phase. $\varepsilon(n)$ is the complex-valued white noise with i.i.d real and imaginary parts, $E[\varepsilon(n)] = 0$, $\mathrm{var}[\varepsilon(n)] = \sigma^2$. The modulus of DSEFT is:

$$\begin{aligned} |X(n,\omega;\lambda)|^2 &= \sum_{k_1=-L}^{L}\sum_{k_2=-L}^{L} [s(k_1)+\varepsilon(k_1)]\cdot[s^*(k_2)+\varepsilon^*(k_2)] \\ &\cdot e^{-\lambda|k_1-n|}\cdot e^{-\lambda|k_2-n|}\cdot e^{-j\omega(k_1-k_2)} \\ &= Y_1(n,\omega;\lambda) + Y_2(n,\omega;\lambda) \end{aligned}$$

$$Y_1(n,\omega;\lambda) = A^2 \sum_{k_1=1}^{n}\sum_{k_2=1}^{n} \mathrm{e}^{j[\phi(k_1)+\phi(k_2)]}\cdot e^{-\lambda|k_1-n|}\cdot e^{-\lambda|k_2-n|}\cdot e^{j\omega(k_1-k_2)} \tag{6}$$

$$\begin{aligned} Y_2(n,\omega;\lambda) = \sum_{k_1=-\infty}^{n}\sum_{k_2=-\infty}^{n} &[s(k_1)\varepsilon^*(k_2) + s(k_2)\varepsilon^*(k_1) \\ &+ \varepsilon(k_1)\varepsilon^*(k_2)]\cdot e^{-\lambda|k_1-n|}\cdot e^{-\lambda|k_2-n|}\cdot e^{-j\omega(k_1-k_2)} \end{aligned} \tag{7}$$

Using the Taylor series of $Y_1(n,\omega;\lambda)$ around time instant n, When $(k_1-n)\to 0, (k_2-n)\to 0$, and the terms whose order is greater than three be neglected, $Y_1(n,\omega;\lambda)$ can be approximated as:

$$Y_1(n,\omega;\lambda) = Y_{10}(n,\omega;\lambda) + Y_{11}(n,\omega;\lambda) + Y_{12}(n,\omega;\lambda)$$

where:

$$Y_{10}(n,\omega;\lambda) = A^2 \sum_{k_1=-L}^{L}\sum_{k_2=-L}^{L} e^{j(k_1-k_2)\phi'(n)}\cdot e^{-\lambda|k_1-n|}\cdot e^{-\lambda|k_2-n|}\cdot e^{-j(k_1-k_2)\omega} \tag{8}$$

$$\begin{aligned} Y_{11}(n,\omega;\lambda) = \lambda^2 A^2 \sum_{k_1=-L}^{L}\sum_{k_2=-L}^{L} &j\cdot\phi^{(2)}(n)\cdot[(k_1-n)^2-(k_2-n)^2]/2 \\ &\cdot e^{j(k_1-k_2)\phi'(n)}\cdot e^{-\lambda|k_1-n|}\cdot e^{-\lambda|k_2-n|}\cdot e^{-j(k_1-k_2)\omega} \end{aligned}$$

$$\begin{aligned} Y_{12}(n,\omega;\lambda) = A^2 \sum_{k_1=-L}^{L}\sum_{k_2=-L}^{L} &j\cdot\phi^{(3)}(n)\left[(k_1-n)^3-(k_2-n)^3\right]/6 \\ &\cdot e^{j(k_1-k_2)\phi'(n)}\cdot e^{-\lambda|k_1-n|}\cdot e^{-\lambda|k_2-n|}\cdot e^{-j(k_1-k_2)\omega} \end{aligned}$$

Generally we would expect the time-frequency representation of a signal peaks about the IF. So the IF estimation of DSEFT is determined by the zero value of the derivative of $Y(n,\omega;\lambda)$. The linearization of the derivative of $Y(n,\omega;\lambda)$ can be written as:

$$\begin{aligned} \left.\frac{\partial\left\{|X(n,\omega;\lambda)|^2\right\}}{\partial\omega}\right|_{\omega=\hat{\omega}_i} &= \left.\frac{\partial Y_{10}(n,\omega;\lambda)}{\partial\omega}\right|_{\omega=\omega_i} \\ &+ \Delta\omega\cdot\left.\frac{\partial^2\{Y_{10}(n,\omega;\lambda)\}}{\partial^2\omega}\right|_{\omega=\omega_i} + \left.\frac{\partial\{Y_{11}(n,\omega;\lambda)\}}{\partial\omega}\right|_{\omega=\omega_i} \\ &+ \left.\frac{\partial\{Y_2(n,\omega;\lambda)\}}{\partial\omega}\right|_{\omega=\hat{\omega}_i} \end{aligned}$$

Note that $\partial Y_{10}(n,\omega)/\partial\omega|_{\omega=\omega_i} = 0$ because of the symmetry of the double-sided exponential window. We get an expression for the bias of DSEFT as an IF estimator,

$$\Delta\omega = -\frac{\dfrac{\partial Y_{11}(n,\omega_i)}{\partial\omega_i} + \dfrac{\partial Y_{12}(n,\omega_i)}{\partial\omega_i} + \dfrac{\partial Y_2(n,\omega_i)}{\partial\omega_i}}{\dfrac{\partial^2 Y_{10}(n,\omega_i)}{\partial^2\omega_i}} \tag{9}$$

where the derivative is calculated at the point $\omega_i = \phi'(n)$, and $\Delta\omega$ is defined as: $\Delta\omega = \omega_i - \hat{\omega}_i$.

The element of (9) is calculated, we get:

$$\frac{\partial Y_{11}(n,\omega_i)}{\partial\omega_i} = 0 \tag{10}$$

$$\frac{\partial Y_{12}(n,\omega_i)}{\partial\omega_i} = \frac{32\cdot A^2\cdot\phi^{(3)}(n)}{\lambda^6} \tag{11}$$

$$E\left(\frac{\partial Y_2(n,\omega_i)}{\partial\omega_i}\right)=0 \tag{12}$$

$$\frac{\partial^2 Y_{10}(n,\omega_i)}{\partial^2\omega_i}=-\frac{16A^2}{\lambda^4} \tag{13}$$

It can be verified that the expectation of $\partial Y_2(n,\omega_i)/\partial\omega_i$ is equal to zero. So the expected value of the IF estimation bias is

$$E[\Delta\omega]=-\frac{2\phi^{(3)}(n)}{\lambda^2} \tag{14}$$

The bias of the estimate depends both on the forgetting factor and the second derivative of the phase.

The MSE of the estimator is given by:

$$E\left[\left|(\Delta\omega)\right|^2\right]=\frac{E\left\{\left[\left(\frac{\partial\{Y_2(n,\omega;\lambda)\}}{\partial\omega}\right)\cdot\overline{\left(\frac{\partial\{Y_2(n,\omega;\lambda)\}}{\partial\omega}\right)}\right]\Bigg|\omega=\hat{\omega}_i\right\}}{\left[\frac{\partial^2 Y_{10}(n,\omega_i)}{\partial^2\omega_i}\right]^2} \tag{15}$$

Substituting equation (13) into (15), we can get the MSE of DSEFT:

$$E\left[\left|(\Delta\omega)\right|^2\right]=\frac{\lambda^3\sigma^2{}_v}{128\cdot A^2} \tag{16}$$

Let SNR be defined as: $SNR=A^2/\sigma^2{}_v$. The relationship of the MSE with SNR and forgetting factor can be written as below:

$$E\left[\left|(\Delta\omega)\right|^2\right]=\frac{\lambda^3}{128\cdot SNR} \tag{17}$$

Equation (17) shows that the MSE of DSEFT IF estimator is an increasing function, while the bias is a decreasing function of forgetting factor. The MSE is determined by the forgetting factor and the SNR level.

As a comparison, the performances of EFT as an IF estimator are presented. The bias is given as,

$$E[\Delta\omega]_{EFT}=\phi^{(2)}(n)/\lambda \tag{18}$$

while the MSE is

$$E\left[\left|(\Delta\omega)\right|^2\right]_{EFT}=\frac{\lambda^3}{16\cdot SNR} \tag{19}$$

4. Simulations

In order to evaluate the proposed algorithm, two kinds of signals are used in the simulations. The first is the signal whose second derivative of the phase increase linearly: $e^{j2\pi\cdot 0.1t^3}$. And the second with its third phase derivative ramps: $e^{j2\pi\cdot 0.01\cdot t^4}$. Fig.1 shows that the bias of the IF estimation using EFT increase linearly, whereas the bias of DSEFT is zero. Fig.2 illustrated the IF estimation error for the second kind of signal, the bias of DSEFT IF estimator increase linearly but slowly, which is much smaller than that of EFT. In both case, DSEFT has much less the bias of IF estimation than EFT. So DSEFT is a more effective IF estimator.

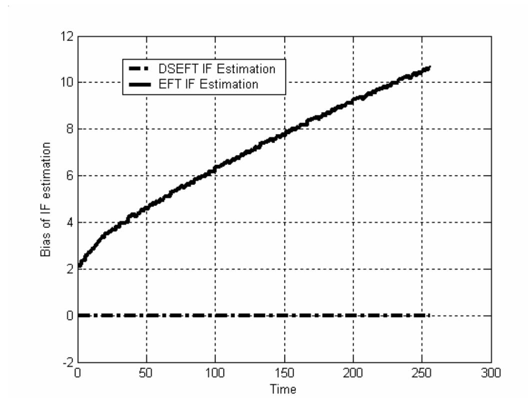

Fig.1. Bias of IF estimation for the signal: $e^{j2\pi\cdot 0.1t^3}$
"__"for EFT, "_." For DSEFT.

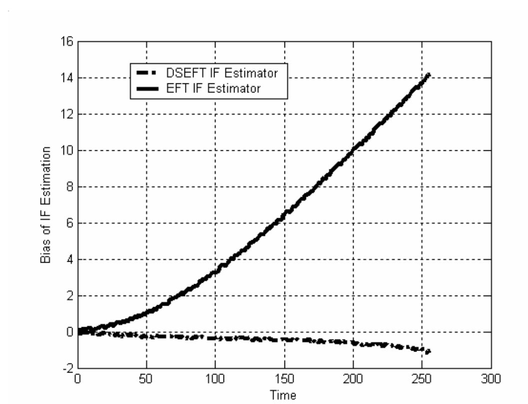

Fig.2 Bias of IF estimation for signal: $e^{j2\pi\cdot 0.01\cdot t^4}$
"__"for EFT, "_." For DSEFT.

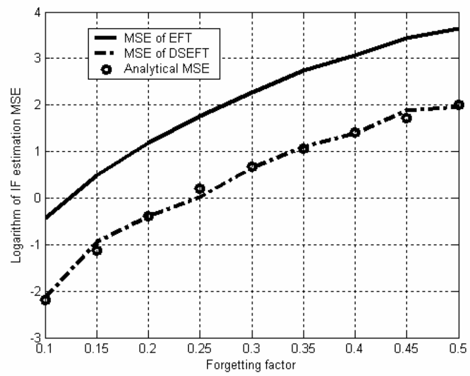

Fig.3. Relationship of MSE with forgetting factor.
"__"for EFT, "_." for DSEFT, "o" for analytical results.

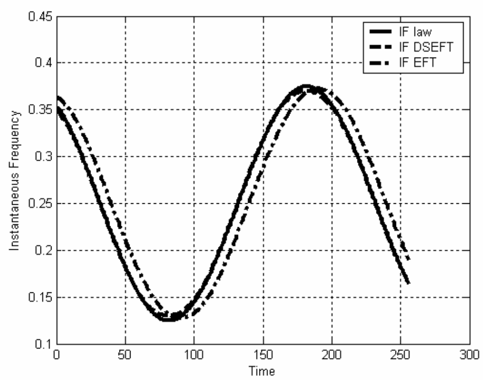

Fig.4 Comparison of IF estimation
"__"for IF law, "_." for EFT and "_ _" for DSEFT

Fig.3 shows the variation of the MSE with forgetting factor, there are three curves representing the logarithms of MSE of the EFT estimator, the DSEFT and the analytical conclusion. According to the analysis, the MSE of the DSEFT at given SNR level is only $1/8$ as great as that of the EFT, a fact which is demonstrated by Fig.3.

In Fig.4, simulation is run to compare the performance of the IF estimators, on a sinusoidal FM signal. From the top down is the IF law of the signal, the IF estimates using DSEFT and EFT. Since the second derivative of phase of the signal can not be neglected, the excursion of IF representation is resulted in EFT, while DSEFT is a good descriptor of IF of non-stationary signals.

5. Conclusions

The algorithm of IF estimation based on DSEFT is proposed. Compared with EFT, this estimator has much less bias and MSE, with the elimination of IF representation excursion. And the analytical expressions for the performance of the DSEFT peak selection in IF estimation has been presented. The validness of the proposed algorithm is demonstrated by the simulations, which shows that DSEFT is an estimator superior to the traditional EFT. The conclusion enables us to properly choose the parameter, such as forgetting factor, of the estimator in IF estimation.

Reference:

[1] S.M. Tretter. Estimation of the Frequency of A Noisy Distribution by linear Regression. IEEE Transactions on Inform. Theory. 1985, 31:823-835.

[2] P.Rao , F.J.Taylor . Estimation of Instantaneous Frequency Using the Discrete Wigner Distribution . IEEE,Electronics letter . 1990, 26(4): 246-248 .

[3] Vladimir Katovnik, L Stankovic . Instantaneous Frequency Estimation Using the Wigner Distribution with Varying and Data-Driven Window length . IEEE Transactions on Signal Processing . 1998 , 46(9): 2315-2325.

[4] L Stankovic , V Katkovic . Algorithm for the Instantaneous Frequency Estimation Using Time-Frequency Distributions with Adaptive Window Width . IEEE Signal Processing letters . 1998. 5(9): 224-227.

[5] L Stankovic , Igor Djurovic . Instantaneous Frequency Estimation by Using the Wigner Distribution and Linear Interpolation . Signal Processing . 2002 . Sigpro 2163 . 1-9.

[6] Tomazic, S., Znidar S. A Fast Recursive STFT Algorithm Electrotechnical Conference, 1996.2: 1025-1028.

[7] Saso Tomazic. On Short-time Fourier Transform with Single-sided Exponential Window. Signal processing, 1996, 55(4): 141-148.