# Design of orthogonal LPTV filters: Application to spread spectrum multiple access

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*Abstract*— **This paper is focused on the design of a set of orthogonal Linear Periodic Time Varying (LPTV) filters and proposes to apply this result to spread spectrum multiuser transmissions. The construction method of such an orthogonal set is presented, emphasizing that only an invertible LPTV filter is required, since the full set is then deduced from this kernel element. Based on this theoretical result, a spread spectrum multiuser transmission is proposed. In order to appreciate the performance of the proposed system, simulations are performed and results are compared to a classical DS-CDMA system.**

## I. Introduction

Linear Periodically Time Varying (**LPTV**) filters are widely used in signal processing applications. Some, explicitly use LPTV filters as for spread spectrum applications[1], transmultiplexing [2], blind channel estimation [3] and spectral scrambling [4]. Others implicitly use LPTV filters as for temporal scrambling (LPTV properties of periodical interleavers) or multirate digital processing such as filterbanks (LPTV nature of interpolation and decimation [5]).

Spreading properties of LPTV filters make them an attractive original tool for multiple access applications and therefore motivated recent works [1],[6]. Such an application meanwhile leads to a double requirement : on one hand it is necessary to design a set of invertible LPTV filters (one LPTV filter is actually assigned to each user) and on the other hand, the orthogonality property must be guaranteed between the set of LPTV filters to circumvent the multiuser interference. With respect to the first requirement, [7] dealt with the design of invertible LPTV filters, emphasizing good spreading properties of this set without considering any orthogonality properties. Similarly, in [6], where a multiple access system is proposed, invertibility is apparently ensured since the LPTV filters are block interleavers (namely permutations), and concerning the orthogonality definition, it is suggested that interleavers are randomly designed for each user.

In this paper, we present a theoretical method for designing a set of orthogonal LPTV filters. In addition, this design method requires only an invertible LPTV filter since the remaining LPTV filters are deduced from this kernel LPTV filter.

In part II, a short review of LPTV filters is presented. Part III proposes the theoretical design of a set of orthogonal filters. This result is applied in part IV to the design of a spread spectrum multiuser system. In part V, the performances of this LPTV multiple access are finally pointed out and compared to a DS-CDMA system.

## II. Review of LPTV filters

In this part, we will present LPTV filters and their invertibility properties.

### A. Notations

Given an integer $N$, we define $W_N^k = e^{-\frac{2i\pi k}{N}}$ as the $N$ roots of unity. Given two integers $(n, N)$, $[n]_N$ stands for the remainder of the Euclidian division of $n$ by $N$. In addition, we define $Int(r)$ as the integer part of given real number $r$.

$X(z)$ stands for the $z$-transform of a digital signal $x(n)$ and is defined by (1). In addition, it is convenient for our theoretical developments to define the vector $\tilde{\mathbf{X}}$ (2).

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad (1)$$

$$\tilde{\mathbf{X}} = \left[ \begin{array}{cccc} X(z) & X(zW_N) & ..... & X(zW_N^{N-1}) \end{array} \right]^{\mathbf{T}} \quad (2)$$

### B. Definition of an LPTV filter

**Definition 1** : An LPTV filter is a filter whose impulse response is an $N$ periodic function. Thus, with $h(n,k)$ the impulse response of the LPTV filter, the output $y(n)$ is expressed by (3) in terms of the input $x(n)$.

$$y(n) = \sum_{k=-\infty}^{\infty} h(n,k)x(n-k), \quad h(n+N,k) = h(n,k) \quad (3)$$

Given an $N$ periodic LPTV filter, many representations, thoroughly listed in [8], are possible. In this framework, we will only handle the modulator representation (Figure 1) defined by $N$ Linear Time Invariant (**LTI**) filters $\{T_p(z)\}$, $(0 \leq p \leq N-1)$ such that $z$-transform $Y(z)$ is obtained from $X(z)$ by relation (4).

$$Y(z) = \sum_{p=0}^{N-1} T_p(zW^p) X(zW^p) \quad (4)$$

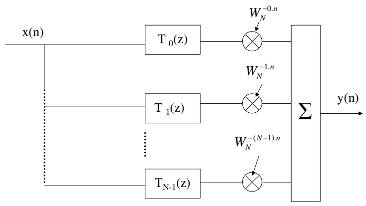


Figure 1 : Modulator representation of an LPTV filter.

Invoking modulator definition of an LPTV filter and vector definition (2), we obtain the matrix formulation (5) for an LPTV filter.

$$\tilde{\mathbf{Y}}=\tilde{\mathbf{T}}\tilde{\mathbf{X}} \quad \text{with} \quad \tilde{\mathbf{T}}_{i,j}=T_{[j-i]_N}(zW^j) \tag{5}$$

It is instructive to note that for an LTI filter $H(z)$, this matrix representation results in a diagonal matrix that will be denoted by $\tilde{\mathbf{D}}^H$ in the following with $\tilde{\mathbf{D}}^H_{i,i} = H(zW^i)$.

*C. Invertibility of an LPTV filter*

**Definition 2** : Given an LPTV filter, with period $N$, defined by its modulator matrix $\tilde{\mathbf{T}}$, this LPTV filter is invertible if there is an LPTV filter with a modulator matrix $\tilde{\mathbf{G}}$ such that relation (6) is fulfilled where $I_N$ stands for the $N$ identity matrix. In other words, the cascade of these two LPTV filters results in the identity operation, cancelling time dependance.

$$\tilde{\mathbf{G}}\tilde{\mathbf{T}}=\mathbf{I}_N \tag{6}$$

Actually, LPTV invertibility leads to a matrix invertibility problem for $z$ belonging to the complex unit circle. By the way, a necessary and sufficient condition is given in [9].

## III. Proposition of a set of orthogonal LPTV filters

In this part, we will first define what we call an orthogonal system. Then, we will describe how to perform a set of orthogonal LPTV filters from only one invertible LPTV filter with modulator matrix $\tilde{\mathbf{T}}^{(0)}$.

*A. Definition of an orthogonal system*

**Definition 3** : Given a domain $E$ as well as $\{A_i\}$, and $\{B_i\}$, $(0 \leq i \leq P-1)$ two sets of $P$ transformations defined on $E$, we say that this couple of sets constitutes a $P$ orthogonal system on $E$ if relation (7) holds where $\bullet$ stands for the composition operation.

$$\forall x \in E, 0 \leq i, j \leq P-1,\ A_i \bullet B_j(x) = \delta(i-j)x \tag{7}$$

*B. Proposition of a set of orthogonal LPTV filters*

Proofs of proposition 1 and 2 are given in Appendix.

**Proposition 1** : Let be an LPTV filter with period $N$ defined by its modulator filters $\left\{T_p^{(0)}(z)\right\}$, $(0 \leq p \leq N-1)$ and an integer $k_0$, the LPTV filter with modulator filters defined by (8) is still an invertible LPTV filter.

$$T_p^{(k_0)}(z) = T^{(0)}_{[p-k_0]_N}(zW_N^{-k_0}) \tag{8}$$

In the following, we will denote $\tilde{\mathbf{T}}^{(k_0)}$ and $\tilde{\mathbf{G}}^{(k_0)}$ as the modulator and inverse modulator matrices of these LPTV filters.

**Proposition 2** : Given a sampling frequency $f_e$ and $0 \leq b \leq 1$, we denote $E_b^{f_e}$ as the set of $f_e$ sampled digital baseband signals with spectrum included in the normalized band $[-\frac{b}{2}; \frac{b}{2}]$. We define $\beta = Int(bN) + 1$ and $\nu = Int(N/\beta)$ where $N$ is an integer. Given an $N$ periodic invertible LPTV filter defined by its modulator matrix $\tilde{\mathbf{T}}^{(0)}$, we define the set of $N$ LPTV filters according to proposition 1 and $H(z)$ an ideal lowpass digital filter with normalized band $[-\frac{b}{2}; \frac{b}{2}]$. Then, the couple of sets of LPTV filters with modulator matrices $\left\{\tilde{\mathbf{T}}^{(i\beta)}\right\}$ and $\left\{\tilde{\mathbf{D}}^H\tilde{\mathbf{G}}^{(i\beta)}\right\}$, for $0 \leq i \leq \nu-1$, constitutes an orthogonal system on the domain $E_b^{f_e}$ according to definition **3**.

## IV. Use of the orthogonal set for spread spectrum multiple access

We will first remind what a multiple access system is from a classical point of view, reminding traditional solutions. Then, using previous results, we propose how to carry out a multiuser system relying on the above considered LPTV filters set.

*A. Principle of spread spectrum multiple access*

Given a set of $P$ users, each conveying a baseband signal with symbol rate $D_S$, a multiuser system is said to be spread spectrum if each user occupies the overall same band $B$ after emission, with $B > PD_S$. The orthogonality property of the system characterizes the possibility of extracting the signal of any user in reception, mitigating interference of the remaining users called multiuser interference.

The most classical solution of such multiuser system is Direct Sequence CDMA (**DS-CDMA**). In this case, user $i$ signal is multiplied by its own code $C^i_j$ for $0 \leq j \leq N-1$ where $N$ is the code's length. Such a code can be seen as a trivial $N$ periodic LPTV filter with impulse response (3) given by $h^i(n,k) = C^i_{[n]_N}$. In the following, we will interest us to an other case of LPTV filters.

*B. Proposed spread spectrum multiple access*

B.1 Parameters of the system

We suppose that each user has a symbol rate $D_S$ and $B$ is the total bandwidth available for the spread spectrum system. $N$ is the period of the kernel LPTV filter. The choice of the latter is not discussed here, but has be done to present good performances for a desynchronized multiuser transmission. Namely, the cascade of the LPTV and its inverse desynchronised LPTV filter must result in a nearly LTI filter (it means that we have for the equivalent modulator filters $|T_p(\omega)| \approx 0$ for $p \neq 0$). This desynchronisation robustness consideration results in the choice of the $N$ periodic uniform interleaver. This chosen LPTV filter is a block interleaver with size $N$ and is composed of a matrix with $P$ lines and $Q$ columns $(PQ = N)$. The input signal is written in columns, and the output signal is obtained by reading in lines. Actually, it was shown that an interleaver is an LPTV filter [6], hence it is possible to compute the corresponding modulator filters.

B.2 Description of the digital system

At the emission side, every oversampled signal $u_i(n)$ is filtered by a digital low pass filter namely a Square Root Cosine (**SRC**) filter with rolloff $\alpha = 0.5$. So we have the assumption that every user has a signal belonging to $E^{LD_S}_{\frac{(1+\alpha)}{L}}$ according to notation of proposition 2, where $L$ is the oversampling factor. According to proposition 2, we can construct an LPTV orthogonal set. We have then the system representation of Figure 2 where $LPTV_i$ is the LPTV filter with modulator matrix $\tilde{\mathbf{T}}^{(i\beta)}$, $\beta = Int(\frac{(1+\alpha)}{L}N) + 1$ and $\nu = Int(\frac{N}{\beta})$. Let us remark that $\nu$ is the maximal number of users without loss of orthogonality.

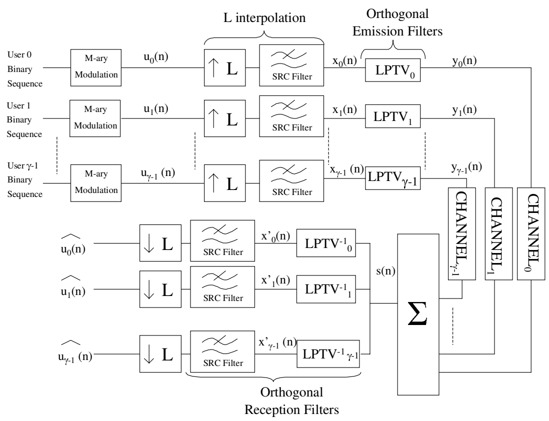


Figure 2 : Architecture of the multiple access system

It is worth to remark that relation (12) in the appendix means that the $k_0$-th LPTV filter is obtained by cascading the LPTV filter of $\mathbf{C}^{k_0}$ modulator matrix and the kernel LPTV filter. According to relation (5) and Figure 1, the LPTV filter corresponds then to a combined modulator and interleaving operation described in Figure 3.

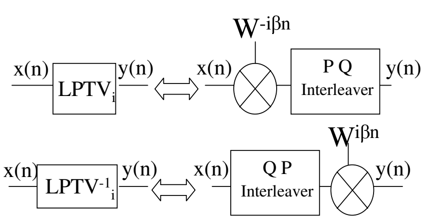


Figure 3 : Equivalent structure for the LPTV filters

## V. Evaluation of the multiuser system performance

In this last part, we present the behavior of the proposed multiuser system with respect to multiuser interferences and we compare it to a DS-CDMA system.

### A. Spread spectrum properties of the proposed multiuser access

Spread spectrum property of any LPTV filter has already been considered in previous works [1],[6],[7]. Here we will only point out the spread spectrum efficiency of the proposed family of orthogonal LPTV filters with the $N = PQ$ periodic uniform interleaver. System of Figure 2 is realized with the following parameters (9).

$$P = 15 \quad Q = 300 \quad \nu = 6 \quad channel_i = 1 \qquad (9)$$

On figure 4, we illustrate the input spectrum of one user before (spectrum of $x_4(n)$) and after (spectrum of $y_4(n)$) the LPTV filtering as well as the total spread spectrum signal (spectrum of $s(n)$).

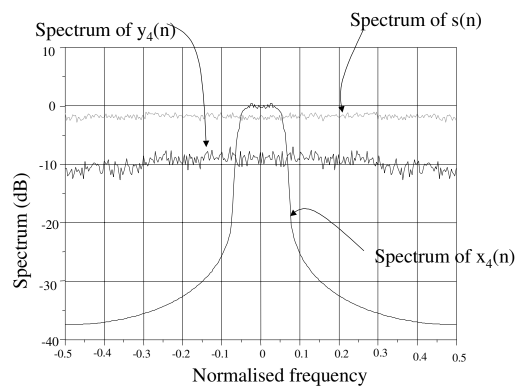


Figure 4 : Spectrum of $x_4(n)$, $y_4(n)$ and $s(n)$

### B. Performance of the system

In this last section, we will consider the performance of our system with respect to the multiuser interference in terms of Bit Error Rate (**BER**). In case all the users are synchronised in reception ($channel_i = 1$ for all $i$) the system is perfectly orthogonal and the BER performances are similar to single user performances. Multiuser interference is actually related to desynchronisation of the users. So, in order to consider the multiuser interference effect, we make the assumption that $channel_0$ equals 1 (the considered user is supposed to be synchronised) and $channel_i$ is a randomly generated delay. The BER are computed and compared to the DS-CDMA system with Gold sequences where these sequences are randomly desynchronised. The parameters for the two systems are the following (10),(11). Figure 5 presents the BER in terms of the number of users.

$$LPTV : L = 31, P = 17, Q = 188, \nu = 20 \qquad (10)$$

$$DS-CDMA : L = 31, polynomials = [45, 75]_{oct} \qquad (11)$$

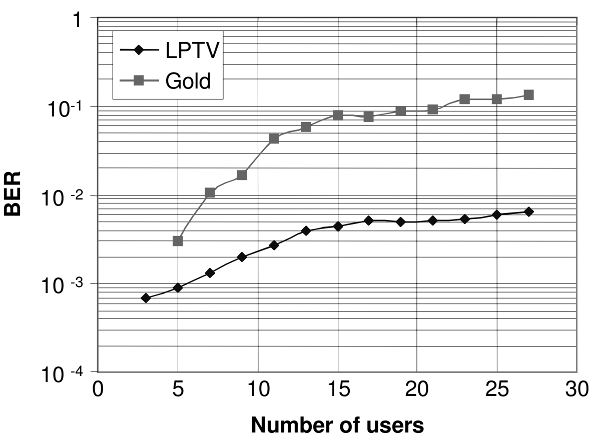


Figure 5 : BER in terms of number of users

## VI. Conclusion

In this paper, we tried to exploit the attractive spreading property of the LPTV filters in order to perform a multiuser system. In spread spectrum multiuser context, a main parameter is the desynchronisation assumption between the users. Actually, users may be either absolutely not or weakly desynchronised, in case of synchronous links, such as down links satellite communication, or they may be randomly desynchronised in case of asynchronous links. Traditional solutions generally presents either good properties under the first assumption, or under the second assumption, but never both. The originality of this work is that the proposed system performs well in both cases. Indeed, the multiuser system relies on a perfect orthogonal system of LPTV filters and in case of perfect synchronisation of the users, multiuser interference is completely circumvented. Such a property is available in DS-CDMA system with Walsh-Hadamard sequences. However, with these sequences the system performance is worse under the assumption of desynchronisation, resulting in a deeply alteration of the BER. Under assumption of random desynchronisation, simulations have shown good performances of our system with respect to DS-CDMA with Gold Sequences that is a desynchronisation suitable solution. Our solution appears to be an efficient solution for any desynchronisation context. In addition, choice of the kernel LPTV filter is not discussed here, but theoretical considerations about this choice are the purpose of a future work.

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## VII. Appendix

### A. Proof of Proposition 1

If we denote $\tilde{\mathbf{T}}^{(k_0)}$ as the modulator matrix of the LPTV filter with modulator filters $\left\{T_p^{(k_0)}(z)\right\}$(8), then we have the matrix relationship (12) where $\mathbf{C}$ is the cyclic permutation matrix of degree 1 whose $(i,j)$ element is defined by $\mathbf{C}_{i,j} = \delta([i-j+1]_N)$.

$$\tilde{\mathbf{T}}^{(k_0)} = \tilde{\mathbf{T}}^{(0)}\mathbf{C}^{k_0} \tag{12}$$

This LPTV filter is invertible. Indeed, if we denote $\tilde{\mathbf{G}}^{(0)}$ the inverse LPTV matrix of $\tilde{\mathbf{T}}^{(0)}$, since $\mathbf{C}^{k_0}\mathbf{C}^{N-k_0} = I_N$ we can easily obtain the inverse matrix of $\tilde{\mathbf{T}}^{(k_0)}$(13). It is relevant to notice that relations (12) and (13) mean that modulator matrix $\tilde{\mathbf{T}}^{(k_0)}$ and $\tilde{\mathbf{G}}^{(k_0)}$ are respectively obtained from $\tilde{\mathbf{T}}^{(0)}$ by column cyclic permutation of degree $k_0$ and from $\tilde{\mathbf{G}}^{(0)}$ by line cyclic permutation of degree $N-k_0$.

$$\tilde{\mathbf{G}}^{(k_0)} = \mathbf{C}^{N-k_0}\tilde{\mathbf{G}}^{(0)} \tag{13}$$

### B. Proof of Proposition 2

Let $(i,j) \in [0;\nu-1]^2$, according to (12) and (13) we have for $\tilde{\mathbf{D}}^H\tilde{\mathbf{G}}^{(j\beta)}\tilde{\mathbf{T}}^{(i\beta)}$ the expression (14).

$$\tilde{\mathbf{D}}^H\tilde{\mathbf{G}}^{(j\beta)}\tilde{\mathbf{T}}^{(i\beta)} = \tilde{\mathbf{D}}^H\mathbf{C}^{(i-j)\beta} \tag{14}$$

Deriving the matrix $\tilde{\mathbf{D}}^H\mathbf{C}^{(i-j)\beta}$, it turns out to be the modulator matrix of an LPTV filter whose only the $(i-j)\beta$-$th$ modulator filter is not nul and equals $H(zW_N^{(j-i)\beta})$. Hence, given $X(z)$ the $z$-transform of an input signal belonging to $E_b^{f_e}$, the output signal $Y(z)$ of the LPTV filter $\tilde{\mathbf{D}}^H\tilde{\mathbf{G}}^{(j\beta)}\tilde{\mathbf{T}}^{(i\beta)}$ has the expression (15). It results in the frequency relation (16) where $f$ stands for the normalized frequency.

$$Y(z) = H(z)X(zW_N^{(i-j)\beta}) \tag{15}$$

$$Y(f) = H(f)X(f - \frac{(i-j)\beta}{N}) \tag{16}$$

From (16) and the assumptions that $H(f)$ is a perfect low pass filter with a normalized band $[-\frac{b}{2};\frac{b}{2}]$ and $X(f)$ is a baseband signal with spectrum included in $[-\frac{b}{2};\frac{b}{2}]$, we get the expected relation (17).

$$Y(f) = \delta(i-j)X(f) \tag{17}$$