ON THE GRUNBAUM COMMUTOR BASED DISCRETE FRACTIONAL FOURIER TRANSFORM

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ABSTRACT

The basis functions of the continuous *fractional Fourier transform* (FRFT) are linear chirp signals that are suitable for time-frequency analysis of signals with chirping time-frequency content. Efforts to develop a discrete computable version of the fractional Fourier transform (DFRFT) have focussed on furnishing a orthogonal set of eigenvectors for the DFT that serve as discrete versions of the Gauss–Hermite functions. Analysis of the DFRFT obtained from Grunbaum's tridiagonal commuter and the kernel associated with it reveals the presence of both amplitude and frequency modulation in contrast to just frequency modulation seen in the continuous case. Furthermore the instantaneous frequency of the basis functions of the DFRFT are sigmoidal rather than linear.

1. INTRODUCTION

The continuous–time Fractional Fourier transform of a signal $x(t)$ is defined via the integral [4]:

$$X_\alpha(t,u) = \mathbf{F}_\alpha(x(t)) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,u)\,dt,$$

where the Kernel of the transform is given by:

$$K_\alpha(t,u) = \sqrt{\frac{1-j\cot\alpha}{2\pi}} \exp\left(j\frac{t^2+u^2}{2}\cot\alpha - jtu\csc\alpha\right).$$

The kernel of the FRFT can be expanded via Mercer's theorem as:

$$K_\alpha(t,u) = \sum_{p=-\infty}^{\infty} e^{-jp\alpha} H_p(t) H_p(u),$$

where $H_p(t)$ corresponds to the k-th order Gauss Hermite function,

$$H_k(t) = \frac{2^{1/4}}{\sqrt{2^k k!}} e^{-\pi t^2} h_k(t),$$

where $h_k(t)$ is the k-th order Hermite polynomial. The FRFT basis functions are linear chirp signals which provides a framework for analysis of signals with linear-FM type time-frequency content. A DFRFT preserving the rotation aspect of the continuous–time FRFT was defined via the fractional power of the DFT matrix [3]:

$$\mathbf{A}_\alpha(\mathbf{x}) = \mathbf{W}^{\frac{2\alpha}{\pi}}(\mathbf{x}) = \sum_{p=0}^{N-1} e^{-jp\alpha}\mathbf{v}_p\mathbf{v}_p^H(\mathbf{x}). \quad (1)$$

Properties of this DFRFT were analyzed in [3], where it was shown to be a rotation in discrete time–frequency space. The expansion in [3], however, is based on eigenvectors of the DFT that are linearly independent but non orthogonal set. Specifically the DFT has 4 distinct eigenvalues and only those that belong to distinct eigenvalues are orthogonal. Since the basis functions of the continuous FRFT are not bandlimited, directly sampling of the kernel will result in aliasing and approaches based on oversampling will result in a non orthogonal basis [7, 9]. Recent efforts towards finding a discrete FRFT have focussed on the problem of furnishing orthogonal eigenvectors for the DFT, that are discrete versions of the Hermite–Gauss functions. One of these approaches based on the Harper matrix $\mathbf{S}$ has been used for constructing a complete orthogonal set of eigenvectors for DFT eigenvectors [2, 10]. Another discrete version of the FRFT based on Kravchuk functions has been explored in [1].

The particular approach towards obtaining the DFT eigenvectors adopted in this paper uses the tridiagonal commuting matrix introduced by Grunbaum [8]. The motivation behind using this approach is that it furnishes a complete basis for the DFT for any N and the tridiagonal commuting matrix in the limit approaches the second-order differential Hermite-Gauss operator [8]. Recently Mugler and Clary modified the Grunbaum tridiagonal incorporating a scaling factor and the resultant eigenvectors very closely resemble the Gauss-Hermite functions [5]. In this paper, we will focus our analysis on the latter and analyze the discrete FRFT obtained from this commutor, study the properties of the transform and the associated basis functions. Specifically we show that the basis functions contain both amplitude and frequency modulation to preserve orthogonality.

2. ON THE GRUNBAUM DFRFT

The motivation behind the commutor matrix approach towards finding the DFT eigenvectors lies in the fact that if two unitary–symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ commute then they share a basis of eigenvectors. If the eigenvalues of the

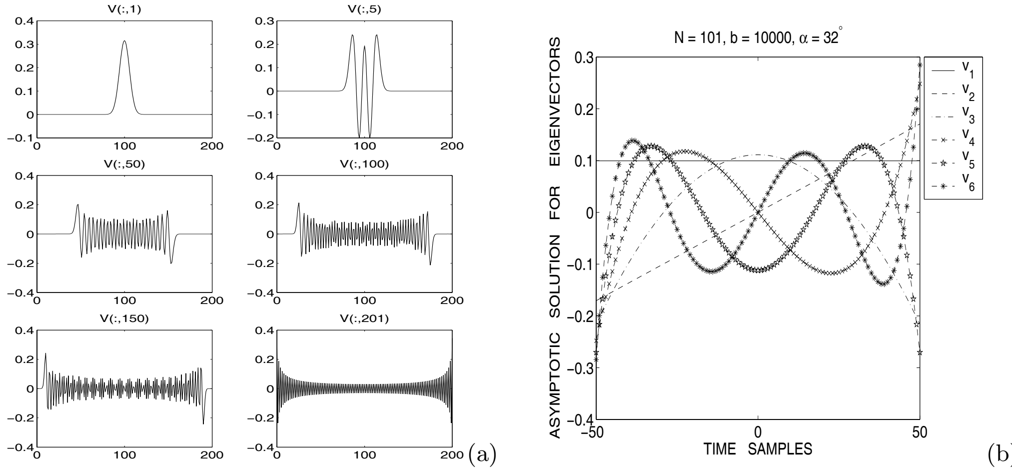

Figure 1: Eigenvectors of the Grunbaum tridiagonal commuting matrix $\mathbf{T}$ for $N = 201, \alpha = 22.5^\circ, b = 1$. Note that the k^{th} eigenvector has $k-1$ zero crossings, (b) asymptotic solution for the eigenvectors $v_k[n], k = 1, 2, 3, 4, 5, 6$.

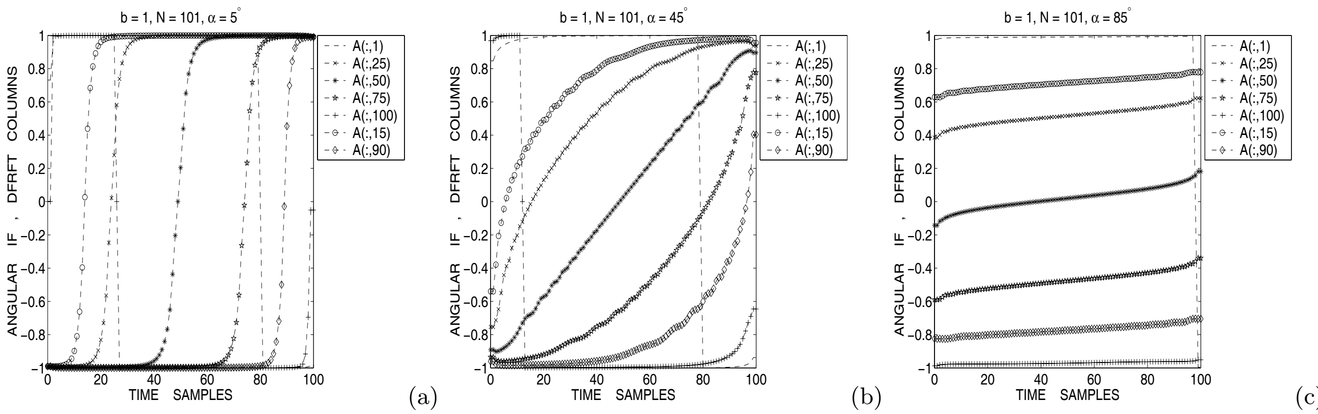

Figure 2: Normalized instantaneous frequency associated with different columns of the DFRFT operator for different angles. The IF of the DFRFT kernel is noticeably sigmoidal rather than linear.

commuting matrix $\mathbf{B}$ are distinct the eigenvectors of the commutor without degeneracy furnish the sought eigenvectors. The tridiagonal commutor of Grunbaum is defined via its diagonal, off-diagonal elements [5]

$$\mathbf{T}_{mn} = \begin{cases} -2\cos(\pi N\tau)\sin(\pi n\tau)\sin(\pi(N-n-1)\tau) \\ \text{if} \quad m = n, \quad 0 \le n \le N-1, \\ \sin(\pi n\tau)\sin(\pi(N-n)\tau) \\ \text{if} \quad m = n+1, n-1, \quad 1 \le n \le N-1, \\ 0, \quad \text{otherwise} \end{cases}$$

where $0 \le \mu \le N-1$ and $\tau = 1/Nb^2$. Next we focus our attention on the centered version of the DFT matrix operator defined via [5]:

$$\{\mathbf{W}_{a,b}\}_{mn} = \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi}{N}(m-a)(n-a)/b^2\right),$$

where the shift parameter $a = \frac{N-1}{2}$. Note that this corresponds to a shifted version of the DFT only when N is odd. This focus is due to the fact that the eigenvalues of the commutor matrix $\mathbf{T}$ for the centered DFT case are both real and unique and furnish the complete orthogonal set of DFT eigenvectors $\mathbf{V}_G$ via [5]: $\mathbf{T} = \mathbf{V}_G\mathbf{\Lambda}_G\mathbf{V}_G^T$. It is also instructive to look at some specific observations regarding the DFRFT that arise out of this expansion in Eq. (1). First, the DFRFT matrix operator is an involution operator of order $m = \text{floor}(\frac{2\pi}{\alpha}) : \mathbf{A}_\alpha^m = \mathbf{I}$. Specifically when $\alpha = \frac{\pi}{2}$ it reduces to the DFT matrix which is $m = 4$-th order involution. The involution property is derived from the eigenvalues of the DFRFT operator and is independent of the eigenvectors. It is also an indicator of the fact that this operator represents a rotation in time–frequency space. The eigenvalues of the DFRFT matrix operator are the roots of unity and when the angle α takes on discrete values $\alpha = \frac{2\pi}{N}p, p = 0, 1, \ldots, N-1$, the trace of the operator vanishes at the zeroes of the Dirichlet kernel:

$$\text{Trace}(\mathbf{A}_\alpha) = D_N(\alpha) = e^{-j\alpha(N-1)/2}\left(\frac{\sin(N\alpha/2)}{\sin(\alpha/2)}\right). \quad (2)$$

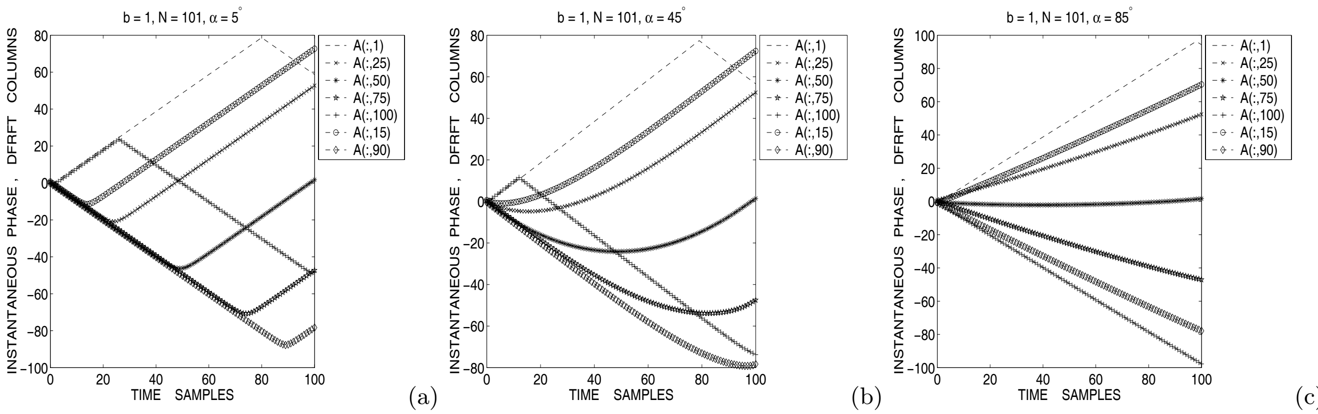

Figure 3: Normalized instantaneous unwrapped phase associated with the kernel of the DFRFT based on the Grunbaum commutor matrix.

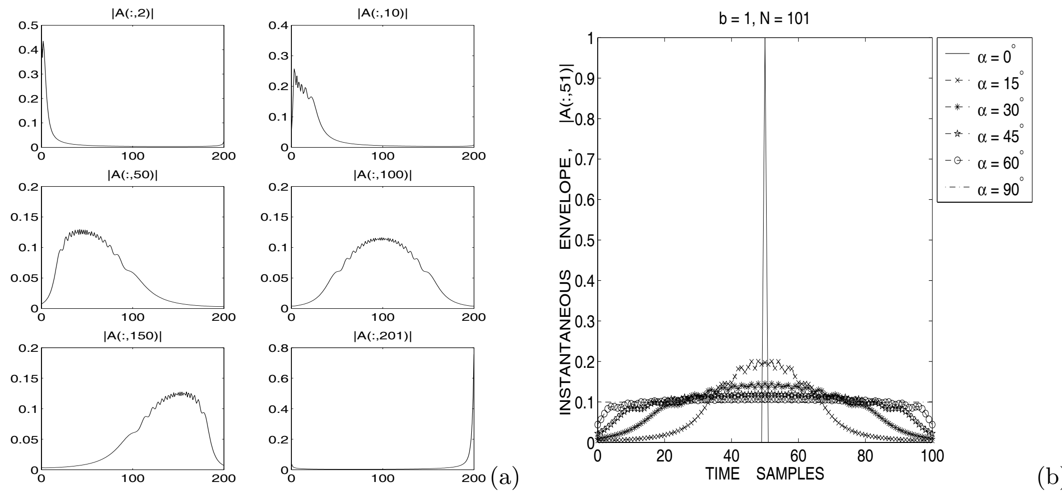

Figure 4: (a) Amplitude of different columns of the discrete FRFT based on the Grunbaum commutor $\mathbf{T}$ for $N = 201, \alpha = 22.5^\circ, b = 1$, (b) instantaneous envelope of a fixed column for different angular parameters describing the increased spread of the envelope with increase in the angular parameter

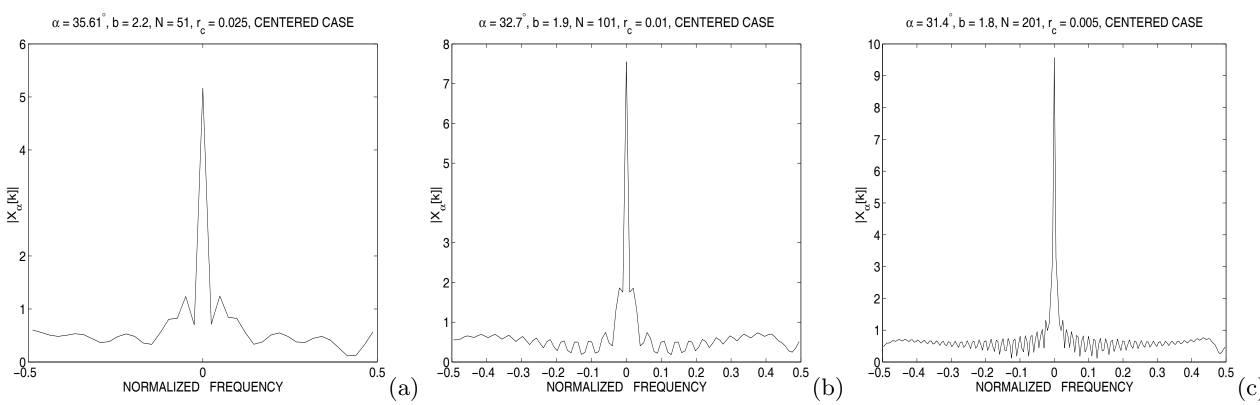

Figure 5: Magnitude of the DFRFT of chirp signals with different chirp rates $r_c = 0.005, 0.025, 0.01$ for specific angles and transform lengths N resulting in a impulse like transform.

When the trace of the DFRFT is actually zero, i.e., $\alpha = \alpha_r = \frac{2\pi r}{N}$ the determinant of the DFRFT operator becomes:

$$\det(\mathbf{A}_\alpha) = \prod_{p=1}^{N} \exp\left(-jp\alpha_r\right) = \pm 1, \quad \alpha_r = \frac{2\pi r}{N} \qquad (3)$$

This fact is important from the perspective of development of fast algorithms for computing the DFRFT because the DFRFT can now be interpreted as a DFT:

$$X_r[k] = \sum_{p=0}^{N-1} \left\{ v_{kp} \sum_{n=0}^{N-1} v_{np} x[n] \right\} W_N^{pr}, \qquad (4)$$

where v_{ij} refers to the (i,j)-th element of the matrix of eigenvectors $\mathbf{V}_G$ of the Grunbaum tridiagonal commutor. which can be computed using the computationally efficient FFT algorithm. The eigenvectors of the Grunbaum tridiagonal commutor $v_k^{(b)}[n]$ for $N = 201, b = 1, \alpha = 22.5°$ are described in Fig. (1)(a). Note that the eigenvector of order k exhibits $k-1$ zero crossings as in the case of the continuous Gauss-Hermite functions. The effect of the dilation parameter on the eigenvectors of $\mathbf{T}$ is illustrated in Fig. (1)(b) for different values of the dilation parameter b. Note that the dilation parameter only affects the eigenvectors and not the eigenvalues of the DFRFT. As the dilation parameter value increases, the spread of the eigenvector $v_1[n]$ increases. Furthermore negative values of dilation parameter b produce the same results as the corresponding positive dilation parameter, i.e., $v_k^{(b)}[n] = v_k^{(-b)}[n]$, indicating a dependence on just $|b|$. These eigenvectors $v_k^{(b)}[n]$ exhibit even or odd symmetry: $v_k^{(b)}[n] = \pm v_k^{(b)}[-n]$ depending on the order k requiring the storage of just half of the N samples for each eigenvector. The eigenvectors of the Grunbaum tridiagonal commutor in particular satisfy a second order difference equation of the form:

$$b_{n+1} v_k^{(b)}[n] + (a_{n+2} - \lambda_k) v_k^{(b)}[n+1] + b_{n+2} v_k^{(b)}[n+2] = 0,$$

where $a_n = \mathbf{T}_{nn}, \ \ 0 \le n \le N-1$ and $b_n = \mathbf{T}_{n,n-1}, \ \ 1 \le n \le N-1$. Fig. (1)(b) describes the effect of a very large dilation parameter b on the eigenvectors $v_2[n], v_3[n]$ and $v_4[n]$ of the Grunbaum tridiagonal commutor $\mathbf{T}$. An important observation that one derives from Fig. (1)(b) is that in the limit of a large dilation parameter the solution to this second-order difference equation $v_k^{(b)}[n]$ approaches a polynomial similar to the way in which the Hermite Gauss functions asymptotically tend to Hermite polynomials:

$$v_k^{(b)}[n] = p_k[n] \psi_k^{(b)}[n], \quad \lim_{b\to\infty} \psi_k^{(b)}[n] = 1. \qquad (5)$$

Specifically the kernel of the DFRFT based on the Grunbaum tridiagonal commutor contains both amplitude modulation and frequency modulation in an effort to preserve orthogonality:

$$K_\alpha[n,k] = A_\alpha[n,k] \exp\left(j\Phi_\alpha[n,k]\right), \qquad (6)$$

where $A_\alpha[n,k]$ is the instantaneous envelope of the kernel and $\Phi_\alpha[n,k]$ is the instantaneous phase of the kernel. As a consequence of this information the DFRFT can be interpreted as an AM–FM transform of the form:

$$X_\alpha[k] = \sum_{n=0}^{N-1} A_\alpha[n,k] \exp\left(j\Phi_\alpha[n,k]\right) x[n] \qquad (7)$$

The AM and FM modulation parts in particular satisfy:

$$\begin{aligned} \lim_{\alpha\to 90°} A_\alpha[n,k] &= \frac{1}{\sqrt{N}}, \quad \lim_{\alpha\to 90°} \Phi_\alpha[n,k] = \frac{2\pi nk}{N} \\ \lim_{\alpha\to 0°} A_\alpha[n,k] &= \delta[n-k]. \end{aligned}$$

Fig. (4)(a) describes the instantaneous envelope of the DFRFT kernel for $\alpha = 22.5°, b = 1, N = 201$. Fig. (4)(b) describes the instantaneous envelope of a specific column of the DFRFT matrix for different angular parameters describing the increasing spread of the envelope from a impulse to a constant. Fig. (2) describes the instantaneous frequency of the columns of the DFRFT where we note that the IF of the kernel is not linear as in the case of the continuous FRFT kernel but rather sigmoidal. Also note that as the angular parameter α approaches $90°$ the IF starts to approach a constant corresponding to the sinusoidal basis functions of the DFT kernel.

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