DUAL-FREQUENCY DUAL-WAVENUMBER CROSS-COHERENCE OF NONSTATIONARY AND INHOMOGENEOUS HARMONIZABLE RANDOM FIELDS

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ABSTRACT

We present a definition of a dual-frequency dual-wavenumber cross-spectrum of two nonstationary and inhomogeneous harmonizable random fields, which is in fact a generalization of the Loève (dual-frequency) cross-spectrum of two random processes. Furthermore, a geometric argument shows that a proper normalization yields a natural measure of cross-coherence, and application of Cauchy-Schwarz' inequality results in a necessary and sufficient condition for full cross-coherence. Finally, estimators of the cross-spectrum and the cross-coherence based on the multitaper approach are suggested, and tested on simulated data.

1. INTRODUCTION

Random functions of both space and time arise naturally in array processing applications, for instance in sonar systems. Such functions are special cases of what is generally called random fields. Often, stationarity and homogeneity is assumed, at least locally. Stationary and homogeneous random fields have been extensively treated in the literature (see e.g. [1] and references therein). However, these assumptions are rather strict, and often not met in practice. In this paper we will study the spectral properties of nonstationary and inhomogeneous random fields.

2. SPECTRAL PROPERTIES

2.1. Spectral representation of random fields

Let $X(\mathbf{r},t)$ be a real valued continuous random field defined on $\mathbf{R}^n \times \mathbf{R}$, where $\mathbf{r}$ is an n-dimensional space variable, and t is a time variable.

The random field $X(\mathbf{r},t)$ is called harmonizable if there exists a representation of the form [1]

$$X(\mathbf{r},t) = \int e^{j2\pi(\mathbf{q}^T\mathbf{r}+ft)}\, d^m\tilde{X}(\mathbf{q},f), \tag{1}$$

where $m = n+1$, and the integration is over $\mathbf{R}^n \times \mathbf{R}$. The measure $d^m\tilde{X}(\mathbf{q},f)$ is the complex valued increment field, or generalized Fourier transform, of the field $X(\mathbf{r},t)$. One could also call it the infinitesimal Fourier-generator of the field $X(\mathbf{r},t)$. Note that for notational convenience, we have introduced the vector variable $\mathbf{q}$, which is related to the wavenumber vector $\mathbf{k}$ by $\mathbf{k} = 2\pi\mathbf{q}$.

2.2. Dual-frequency dual-wavenumber cross-spectra

Now, consider two random fields $X(\mathbf{r},t)$ and $Y(\mathbf{r},t)$. Assume temporarily that the corresponding increment fields are orthogonal, i.e. [2]

$$\mathbb{E}d^m\tilde{X}^*(\mathbf{q}-\boldsymbol{\kappa},f-\nu)d^m\tilde{Y}(\mathbf{q},f) = S_{XY}(\mathbf{q}\,;f)\delta(\nu)\delta(\boldsymbol{\kappa})\,d^n\mathbf{q}d^n\boldsymbol{\kappa}df\,d\nu, \tag{2}$$

where $\delta(\cdot)$ is Dirac's delta function. In this case we obtain the well known stationary and homogeneous result that the spatio-temporal cross-correlation function $\mathbb{E}X(\mathbf{r},t)Y(\mathbf{r}+\boldsymbol{\rho},t+\tau)$ is a function of $\boldsymbol{\rho}$ and τ only:

$$\mathbb{E}X(\mathbf{r},t)Y(\mathbf{r}+\boldsymbol{\rho},t+\tau) = M_{XY}(\boldsymbol{\rho},\tau) = \int e^{j2\pi(\mathbf{q}^T\boldsymbol{\rho}+f\tau)}S_{XY}(\mathbf{q}\,;f)\,d\mathbf{q}df, \tag{3}$$

and we now recognize $S_{XY}(\mathbf{q}\,;f)$ as the ordinary frequency-wavenumber cross-spectrum.

In general, however, the increment fields are nonorthogonal, i.e.

$$\mathbb{E}d^m\tilde{X}^*(\mathbf{q}-\boldsymbol{\kappa},f-\nu)d^m\tilde{Y}(\mathbf{q},f) = S_{XY}(\boldsymbol{\kappa},\mathbf{q};\nu,f)\,d^n\mathbf{q}d^n\boldsymbol{\kappa}d\nu df, \tag{4}$$

where $S_{XY}(\boldsymbol{\kappa},\mathbf{q};\nu,f)$ is some complex valued function of $\boldsymbol{\kappa},\mathbf{q},\nu$ and f. In this case, the spatio-temporal cross-correlation function assumes the form

$$M_{XY}(\mathbf{r},\boldsymbol{\rho};t,\tau) \triangleq \mathbb{E}X(\mathbf{r},t)Y(\mathbf{r}+\boldsymbol{\rho},t+\tau) = \iint e^{j2\pi(\boldsymbol{\kappa}^T\mathbf{r}+\mathbf{q}^T\boldsymbol{\rho}+\nu t+f\tau)} \cdot \mathbb{E}d^m\tilde{X}^*(\mathbf{q}-\boldsymbol{\kappa},f-\nu)d^m\tilde{Y}(\mathbf{q},f) \tag{5}$$

which may be viewed as a spectral representation of the cross-correlation function $M_{XY}(\mathbf{r},\boldsymbol{\rho};t,\tau)$. Note that this

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is in fact a generalization of the Loève cross-spectrum [3] of two nonstationary random processes.

Correlation among different frequency components is responsible for nonstationarities, while correlation among different wavenumber components results in inhomogeneities. Now, we may regard $S_{XY}(\boldsymbol{\kappa}, \mathbf{q}; \nu, f)$ as a dual-frequency dual-wavenumber cross-spectrum. The variables $\mathbf{q}$ and f are *global* quantities, while $\boldsymbol{\kappa}$ and ν are *local* quantities. This choice of variables is attractive due to the fact that stationary and homogeneous processes will have nonzero values only on the *stationary and homogeneous manifold* $(\boldsymbol{\kappa}, \nu) = (\mathbf{0}, 0)$, as can be seen from (2). With this particular choice of variables, the stationary and homogeneous manifold coincides with the global coordinate axes.

2.3. Dual-frequency dual-wavenumber cross-coherence

The concept of coherence is very important in many applications. Various definitions of a quantity measuring the degree of coherence can be found in the literature. The value of such a measure should preferably be bounded between zero and one. We will in this section derive a dual-frequency dual-wavenumber cross-coherence function by following a procedure analogous to the derivation of a dual-frequency coherence function for nonstationary random processes in [4].

A meaningful way of defining a cross-coherence function of two nonstationary and inhomogeneous random fields $X(\mathbf{r}, t)$ and $Y(\mathbf{r}, t)$ can be obtained by recognizing the fact that the dual-frequency dual-wavenumber cross-spectrum $S_{XY}(\boldsymbol{\kappa}, \mathbf{q}; \nu, f)$ can be expressed as a Hilbert space inner product

$$S_{XY}(\boldsymbol{\kappa}, \mathbf{q}; \nu, f) d^n\boldsymbol{\kappa} d^n\mathbf{q} d\nu df = \left\langle d^m\tilde{Y}(\mathbf{q}, f), d^m\tilde{X}(\mathbf{q} - \boldsymbol{\kappa}, f - \nu) \right\rangle, \quad (6)$$

where we have defined the Hilbert space inner product between two complex valued stochastic variables Z and W by $\langle Z, W \rangle \triangleq \mathbb{E}ZW^*$. A normalized version of $S_{XY}(\boldsymbol{\kappa}, \mathbf{q}; \nu, f)$ is called for since the concept of coherence is related to the phase of the Fourier generators $d^m\tilde{X}(\mathbf{q}, f)$ and $d^m\tilde{Y}(\mathbf{q}, f)$, rather than the magnitude.

Now, associated with any inner product $\langle Z, W \rangle$, there is an angle ψ between Z and W defined by

$$\cos\psi = \frac{\langle Z, W \rangle}{\sqrt{\langle Z, Z \rangle \langle W, W \rangle}}.$$

We can argue that $\cos\psi$ is a reasonable measure of coherence since intuitively a high value of $S_{XY}(\boldsymbol{\kappa}, \mathbf{q}; \nu, f)$ corresponds to high coherence, and the inner product (6) is maximized when ψ is minimized. This justifies the following definition of a dual-frequency dual-wavenumber cross-coherence function

$$\begin{aligned}\rho^2_{XY}(\boldsymbol{\kappa}, \mathbf{q}; \nu, f) &\triangleq \cos^2\psi(\boldsymbol{\kappa}, \mathbf{q}; \nu, f) \\ &= \frac{\left|\mathbb{E}d^m\tilde{X}^*(\mathbf{q} - \boldsymbol{\kappa}, f - \nu) d^m\tilde{Y}(\mathbf{q}, f)\right|^2}{\mathbb{E}\left|d^m\tilde{X}(\mathbf{q} - \boldsymbol{\kappa}, f - \nu)\right|^2 \mathbb{E}\left|d^m\tilde{Y}(\mathbf{q}, f)\right|^2} \\ &= \frac{|S_{XY}(\boldsymbol{\kappa}, \mathbf{q}; \nu, f)|^2}{S_{XX}(\mathbf{0}, \mathbf{q} - \boldsymbol{\kappa}; 0, f - \nu) S_{YY}(\mathbf{0}, \mathbf{q}; 0, f)}.\end{aligned} \quad (7)$$

One can easily show that $0 \leq \rho^2_{XY}(\boldsymbol{\kappa}, \mathbf{q}; \nu, f) \leq 1$ by employing Cauchy-Schwarz' inequality. Furthermore, it follows that we get full cross-coherence $\rho^2_{XY}(\boldsymbol{\kappa}, \mathbf{q}; \nu, f) = 1$ if and only if

$$d^m\tilde{X}(\mathbf{q} - \boldsymbol{\kappa}, f - \nu) = \gamma\, d^m\tilde{Y}(\mathbf{q}, f) \quad (8)$$

for some $\gamma \in \mathbf{R}$. Now, if we decompose the complex valued increment fields as $d^m\tilde{X}(\mathbf{q}, f) \triangleq |d^m\tilde{X}(\mathbf{q}, f)| \exp\{j\phi_X(\mathbf{q}, f)\}$ and $d^m\tilde{Y}(\mathbf{q}, f) \triangleq |d^m\tilde{Y}(\mathbf{q}, f)| \exp\{j\phi_Y(\mathbf{q}, f)\}$, respectively, we arrive at the fundamental result that we have full cross-coherence at $(\boldsymbol{\kappa}, \mathbf{q}; \nu, f)$ if and only if

$$\phi_X(\mathbf{q} - \boldsymbol{\kappa}, f - \nu) = \phi_Y(\mathbf{q}, f) + k\pi \qquad k \in \mathbf{Z}, \quad (9)$$

that is, full cross-coherence is equivalent to the random variables corresponding to the phase of the components $d^m\tilde{X}(\mathbf{q} - \boldsymbol{\kappa}, f - \nu)$ and $d^m\tilde{Y}(\mathbf{q}, f)$, respectively, being identical modulo π. This result is the main contribution of this work.

3. ESTIMATION OF DUAL-FREQUENCY DUAL-WAVENUMBER CROSS-SPECTRA

In this section we will look at the special case where we have a one-dimensional space variable r. Generalization to two or three space dimensions is straightforward.

Assume that we have sampled the random field at spatial positions $m = 0, 1, \ldots, N_r - 1$ and temporal instants $n = 0, 1, \ldots, N_t - 1$. Thus, if the field is harmonizable, the samples $x[m, n] = X(m, n)$ have the spectral representation

$$x[m, n] = \int_{-f_N}^{f_N} \int_{-f_N}^{f_N} e^{j2\pi(qm + fn)}\, d^2\tilde{X}(q, f), \quad (10)$$

where $f_N = 1/2$ is the Nyquist frequency.

For a given bandwidth W, it is well known that a statistically stable estimate of the Loéve cross-spectrum (in the random process case), is Thomson's high-resolution multitaper estimate [5]. This estimate extends naturally to random fields. The eigencoefficients of the high-resolution expansion in this case become [6]

$$\tilde{x}_{k_r, k_t}(q, f) = \sum_{m=0}^{N_r - 1} \sum_{n=0}^{N_t - 1} x[m, n] \cdot v^{k_r}_{N_r, W_r}[m] v^{k_t}_{N_t, W_t}[n] e^{-j2\pi(qm + fn)} \quad (11)$$

where $v^k_{N,W}[l]$ is the kth discrete prolate spheroidal sequence of length N and bandwidth W [7]. Analogous to the multitaper estimate of the Loève cross-spectrum [8], we may form an estimate of $S_{XY}(\kappa, q; \nu, f)$ by averaging over products of the eigencoefficients $\tilde{x}^*_{k_r,k_t}(q-\kappa, f-\nu)$ and $\tilde{y}_{k_r,k_t}(q, f)$

$$\widehat{S_{XY}}(\kappa, q; \nu, f) \triangleq \frac{1}{K_r K_t} \sum_{k_r=0}^{K_r-1} \sum_{k_t=0}^{K_t-1} \tilde{x}^*_{k_r,k_t}(q-\kappa, f-\nu) \tilde{y}_{k_r,k_t}(q, f), \quad (12)$$

where $K_r = \lfloor 2N_r W_r \rfloor$ and $K_t = \lfloor 2N_t W_t \rfloor$. Similarly, we estimate $\rho^2_{XY}(\kappa, q; \nu, f)$ by

$$\widehat{\rho^2_{XY}}(\kappa, q; \nu, f) \triangleq \frac{\left|\widehat{S_{XY}}(\kappa, q; \nu, f)\right|^2}{\widehat{S_{XX}}(q-\kappa, q-\kappa; f-\nu, f-\nu)\widehat{S_{YY}}(q, q; f, f)}. \quad (13)$$

Analysis of the statistical properties of the estimators in (12) and (13) involves generalization of the results in [9] and [5]. However this is outside the scope of the present paper. For this work, it suffices to observe that the multitaper approach stabilizes a naïve estimator based on (single-)tapered FFTs, since it amounts to averaging over estimates that are statistically uncorrelated due to the orthogonality of the discrete prolate spheroidal sequences.

4. NUMERICAL SIMULATIONS

In this section, we will demonstrate some of the properties of the dual-frequency dual-wavenumber cross-coherence by numerical examples. For simplicity, we only consider one spatial dimension, such that the wavenumber vector $\mathbf{q}$ reduces to a scalar q.

4.1. The stationary and homogeneous case

Stationarity and homogeneity corresponds to uncorrelated frequency and wavenumber components. Thus, we expect the dual-frequency dual-wavenumber cross-coherence of a white stationary random field with itself (i.e. the auto-coherence) to be exactly one on the stationary and homogeneous manifold $(\kappa, \nu) = (0, 0)$, and zero elsewhere.

Fig. 1 shows two slices in the dual-frequency direction of the estimated dual-frequency dual-wavenumber auto-coherence of a white Gaussian stationary random field with mean value zero and unit variance. In the estimation, we have used sample sizes $N_r = 16$ and $N_t = 128$, and time-bandwidth products $N_r W_r = 3$ and $N_t W_t = 10$, respectively. In the left panel, where $\kappa \neq 0$, the estimated auto-coherence is close to zero for all frequency pairs (ν, f), with a maximum value of 0.07. The true value is exactly zero for

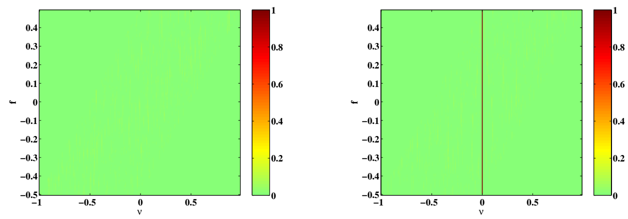

Fig. 1. Slices in the dual-frequency direction of the estimated dual-frequency dual-wavenumber auto-coherence $\widehat{\rho^2_{XX}}(\kappa, q; \nu, f)$ of a white Gaussian stationary random field. Left panel: $q = -0.25$ and $\kappa = -0.6875$. Right panel: $q = -0.25$ and $\kappa = 0$.

all (ν, f). In the right panel, on the other hand, where $\kappa = 0$, we get an estimated auto-coherence equal to one along the line $\nu = 0$, which is the part of the slice that resides in the stationary and homogeneous manifold. Elsewhere, the estimated auto-coherence is close to zero, which is as expected from Eq. (2).

4.2. The nonstationary and inhomogeneous case

We have shown that nonstationarities and inhomogeneities manifest themselves as nonzero dual-frequency dual-wavenumber cross-coherence outside the stationary and homogeneous manifold, where $(\kappa, \nu) \neq (0, 0)$. To demonstrate this effect, let us assume that a chirp-like acoustic signal

$$X(r, t) = \cos(2\pi\beta t^2), \quad (14)$$

where $r = 0, 1, \ldots, N_r - 1$ and $t = 0, 1, \ldots, N_t - 1$, is transmitted through a body of water. The signal is reflected, and arrives at a hydrophone array with some nonzero incident angle. The received signal can then be modeled as

$$Y(r, t) = X(r, t - \tau(r)) + N(r, t), \quad (15)$$

where $\tau(r)$ is the delay with respect to the transmitted signal at the hydrophone in position r, and $N(r, t)$ is some noise field. In Figure 2, we show the estimated dual-frequency dual-wavenumber cross-coherence between $X(r, t)$ and $Y(r, t)$ for $\beta = 0.25/N_t$. The noise field $N(r, t)$ is a zero mean stationary Gaussian field with variance 0.1, and we have used sample sizes $N_r = 8$ and $N_t = 256$, and time-bandwidth products of $N_r W_r = 3$ and $N_t W_t = 4$. The hydrophone array is assumed to be linear, such that $\tau(r + 1) - \tau(r) = \Delta\tau = 2\ \forall r$.

Fig. 2 shows four slices in the dual-frequency direction of the estimated dual-frequency dual-wavenumber cross-coherence of $X(r, t)$ and $Y(r, t)$ in (14) and (15). Notice that in all four slices, we get significantly nonzero estimated cross-coherence around two points on the line $\nu = 0$, and around two points on the line $\nu = 2f$. From this figure, we may

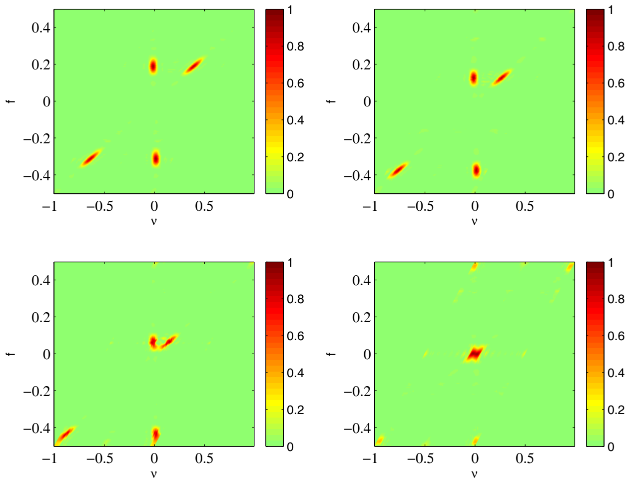

Fig. 2. Slices in the dual-frequency direction of the estimated dual-frequency dual-wavenumber cross-coherence $\widehat{\rho^2_{XY}}(\kappa, q; \nu, f)$ of $X(r,t)$ and $Y(r,t)$ in (14) and (15), respectively. Top left: $q = -0.125$ and $\kappa = -0.375$. Top right: $q = -0.125$ and $\kappa = -0.25$. Bottom left: $q = -0.125$ and $\kappa = -0.125$. Bottom right: $q = -0.125$ and $\kappa = 0$.

observe two important facts. Firstly, all four slices contain non-zero values. If the process had been homogeneous, only the bottom right slice, where $\kappa = 0$, would have contained non-zero values. Thus, we may conclude that the process is inhomogeneous. Secondly, all slices have nonzero values outside the line $\nu = 0$. This means that the process is nonstationary. Thus, we have demonstrated the effect that nonstationarities and inhomogeneities manifest themselves as nonzero values outside of the stationary and homogeneous manifold. Further analysis of this particular example is outside the scope of this paper.

5. CONCLUSIONS

We have in this paper defined a dual-frequency dual-wavenumber cross-spectrum of two nonstationary and inhomogeneous random fields, which is a generalization of the Loève cross-spectrum of two nonstationary random processes.

Interpreting the cross-spectrum as a Hilbert space inner product between the increment fields, or infinitesimal Fourier generators, of the two random fields yielded a proper normalization of the cross-spectrum. This normalization was then interpreted as a measure of cross-coherence.

We have in this paper shown that a necessary and sufficient condition for full dual-frequency dual-wavenumber cross-coherence between two random fields at given frequency-wavenumber pairs, is that the random variables corresponding to the phase of the increment fields at the respective frequency-wavenumber pairs are identical. This result yields fundamental insight into the nature of the concept of coherence, which intuitively is connected to the covariation of the phase of two harmonic signals.

Finally, based on Thomson's multitaper approach, we have suggested and implemented estimators for the proposed cross-spectra and cross-coherence. Numerical examples demonstrated that nonstationarities and inhomogeneities show up as nonzero values of the dual-frequency dual-wavenumber cross-coherence outside of the stationary and homogeneous manifold. Thus, the suggested coherence measure provides valuable insight into systems that are inherently nonstationary and inhomogeneous, including linear time-variant and dispersive systems commonly encountered in array processing applications.

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