ROBUST POLYNOMIAL WIGNER-VILLE DISTRIBUTION FOR THE ANALYSIS OF POLYNOMIAL PHASE SIGNALS IN α-STABLE NOISE

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ABSTRACT

The polynomial Wigner-Ville Distribution (PWVD) is the most suitable Time-Frequency Representation (TFR) of non-stationary signals which can be modelled by polynomial phase signals. It is a very efficient signal analysis tool both in the case of additive and multiplicative Gaussian noise. However in many real life applications, noise can be non Gaussian. Hence the PWVD fails to produce satisfactory results. On the other hand, it is well known that Fractional Lower-Order Statistics (FLOS) are signal processing tools that tend to resist outliers observed in signals. In this paper, we propose a new robust FLOS-based Polynomial Wigner-Ville Distribution which we call the Fractional Lower-Order Polynomial Wigner-Ville Distibution (FLOPWVD). The latter is able to reveal the instantaneous frequency of the high order polynomial phase signal (PPS) in the presence of impulsive noise modelled by α-stable distribution. This new Time-Frequency Distribution outperforms the standard PWVD as will be shown in simulation results.

1. INTRODUCTION

In many fields of engineering such as radar, sonar and telecommunications, the signals processed at the transmitter/receiver are non stationary and can be modelled by constant amplitude polynomial phase signal (PPS) with constant or slowly time-varying amplitude [1]. In the case of PPS having polynomials of degree less or equal to 2, the Wigner-Ville Distribution (WVD) and its smoothed variants provide the optimal energy concentration about the instantaneous frequency (IF) in the time-frequency plane. However, when the polynomial phase signals have a degree higher than 2, the WVD does not give optimal energy concentration resulting in a smeared spectral representation [2]. Hence, in order to deal with this limitation, The Polynomial Wigner-Ville Distribution (PWVD) has been proposed as an adequate Time-frequency distribution for PPS with degree higher than 2 yielding an optimal energy concentration around the IF [2].

In practice, the signal under consideration may be subjected to additive noise which is generally assumed to be Gaussian. Several papers have considered this case as in [3, 4, 5]. However, the assumption of Gaussianity is not valid in some other situations such as those involving atmospheric or underwater acoustic noise which displays impulsive characteristics [6] with heavy-tailed distributions that degrade significantly the signal representation in the time-frequency plane. Impulsive noise can be modelled by α-stable random process. The fact that α-stable random variables with $\alpha < 2$ have infinite second moment means that many techniques based on the Gaussian case will not apply, and therefore, we must consider other alternatives to mitigate the consequence of the non-Gaussian noise on the time-frequency distribution.

In this paper, we extend the work proposed in [7] to address the problem of the time-frequency representation of polynomial phase signals corrupted by additive impulsive noise using Fractional Lower-Order Statistics (FLOS) introduced in [8] which handle robustly the presence of heavy-tailed noise in the data.

This paper is organized as follows. In section 2, we briefly review the different Time-Frequency Distributions (TFD's) related to analysis of polynomial phase signals. Then in sections 3 and 4, we examine the impulsive noise, its characteristics and effect on TFD's respectively. In section 5, we discuss the different TFD's proposed for non-stationary signals corrupted by additive impulsive noise. And we propose a new Fractional Lower Order based Polynomial Wigner-Ville Distribution (FLOPWVD) for PPS signals of order greater or equal to 2. Some simulation examples are presented in section 6. And concluding remarks are given in Section 7.

2. THE POLYNOMIAL PHASE WIGNER-VILLE DISTRIBUTION

The constant amplitude polynomial phase signal of order p is given by

$$z(t) = A\exp\{j\phi(t)\} = \exp\left\{j\sum_{i=0}^{p} a_i t^i\right\} \tag{1}$$

where A is the amplitude of the signal, the a_i's $(i = 0, \ldots, p)$ are real coefficients, and $t \in [0, T]$.

The instantaneous frequency (IF) of the signal given in (1) is defined as

$$f_i(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt} = \frac{1}{2\pi}\sum_{i=1}^{p} i\ a_i\ t^{i-1}. \tag{2}$$

It has been shown that for polynomial phase signals of order $p \leq 2$ the WVD has the optimal energy concentration on the time-frequency plane [9]. However when the order p is greater than 2 the WVD exhibits artifacts that alter the IF estimation. In order to effectively overcome this limitation, the polynomial Wigner-Ville distribution for a signal $x(t)$ has been defined by [2]

$$W_x^q(t, f) = \int_{-\infty}^{\infty} K_x^q(t, \tau)\, e^{-j2\pi f\tau} d\tau \tag{3}$$

where the kernel $K_{q,x}(t,\tau)$ is defined as

$$K_x^q(t,\tau) = \prod_{k=0}^{q/2} [x(t+t_k\tau)]^{\gamma_k} \, [x^*(t+t_{-k}\tau)]^{-\gamma_{-k}} \quad (4)$$

where q is an even integer which corresponds to the order the PWVD with ($q \geqslant p$). The procedure to compute the coefficients t_k and γ_k for a fixed value of p and q is outlined in [10], [2], and [11]. The PWVD transforms the polynomial phase signal into sinusoids which exhibit delta functions around the signal IF when fourier transformed. The WVD is a special case of the PWVD which can be obtained when the parameter $q = 2$, $t_0 = 0$, $t_1 = -t_{-1} = 1/2$, $\gamma_0 = 0$, and $\gamma_1 = -\gamma_{-1} = 1$. Note that the realness of the PWVD implies that $t_k = -t_k$. A numerical example given in [11], proposes a sixth-order PWVD kernel yielding artifact-free representation with a small order of interpolation.

3. α STABLE NOISE DISTRIBUTIONS

There exists many physical processes generating interference containing noise components that are impulsive in nature (e.g., atmospheric noise in radio links; and radar reflections from ocean waves [6], and reflections from large, flat surfaces including buildings and vehicles). The amplitude distributions of such returns are not Gaussian, and tend to produce large-amplitude excursions and occasional bursts of outlying observations. A variety of impulsive noise models can be found in literature. The most commonly used so far is the Middleton's model [12]. Recently, a more flexible model has been suggested for impulsive noise known as the α-stable modelling whose distribution is defined by its characteristic function [8]

$$\Psi(t) = \exp\{j\mu t - \gamma\,|t|^{\alpha}\,[1 + j\beta sign(t)\kappa(t,\alpha)]\} \quad (5)$$

where

$$\kappa(t,\alpha) = \begin{cases} \tan\frac{\alpha\pi}{2}, & \alpha \neq 1 \\ \frac{2}{\pi}\ln|t|, & \alpha = 1 \end{cases}$$

The stable distribution is completely characterized by the parameters α ($0 < \alpha \leq 2$) named the characteristic exponent, β: the symmetry parameter (the distribution is symmetric α-stable SαS when $\beta = 0$ as shown in figure 1), γ is the dispersion ($\gamma > 0$), and μ is the location parameter ($-\infty < \mu < \infty$). The characteristic exponent determines the shape of the distribution. The smaller α is, the heavier the tails of the alpha stable density. We should also note that for $\alpha = 2$ the distribution coincides with the Gaussian density. The dispersion parameter γ determines the spread of the distribution around its location parameter μ in the same way that the variance of a Gaussian distribution determines the spread around the mean [8]. For α-stable processes only the moments of order $m < \alpha$ exist. The variance of α-stable random variables is infinite for $\alpha < 2$, and the mean becomes infinite when $\alpha < 1$. Hence, estimation methods based on second order statistics of the data cannot be applied.

4. EFFECT OF IMPULSIVE NOISE ON TIME-FREQUENCY REPRESENTATIONS

Let us consider the noisy signal $x(t)$ given by

$$x(t) = z(t) + \nu(t) \quad (6)$$

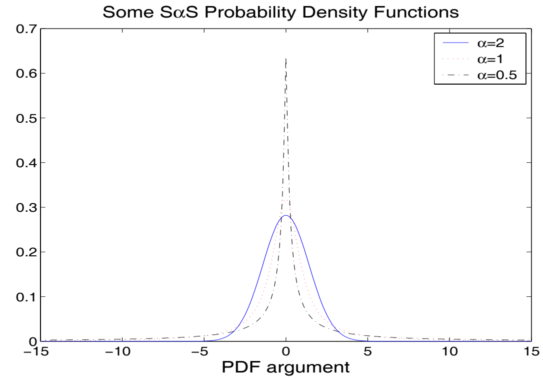

Fig. 1. Symmetric α-stable PDFs with $\mu = 0, \gamma = 1, \beta = 0$ and different characteristic exponents

where $z(t)$ is a polynomial phase signal defined in (1) and $\nu(t)$ is an SαS noise. The impulsive noise profoundly degrades TFRs, although the type of degradation is dependent on the type of TFRs being used. Additive as well as multiplicative Gaussian noise influence on the Time-frequency distributions has been considered in literature by many authors such as in [3],[4], [5]. However, it is common for these standard distributions (WVD, PWVD,...) to produce poor results in an impulsive noise environment as shown in figure 2.

5. THE FRACTIONAL LOWER-ORDER POLYNOMIAL WIGNER-VILLE DISTRIBUTION

In order to overcome the problem of unbounded moments of signals in the presence of α-stable noise, the authors in [13] introduced the Fractional Lower Order Covariance (FLOC) as a new measure of similarity (or difference) of two α-stable processes. The FLOC is defined by

$$FLOC_x^a(t,\tau) = E\{x^{\langle a\rangle}(t+\tau/2)\, x^{-\langle a\rangle}(t-\tau/2)\} \quad (7)$$

The FLOC has finite mean and variance when $0 < a < \alpha/2$, and $(\cdot)^{\langle a\rangle}$ is the called the a^{th}-order phased fractional lower-order moment (PFLOM) operator defined as [14]

$$x^{\langle a\rangle} = |x|^{a+1}/x^* \qquad \text{with } 0 \leq a \leq 1 \quad (8)$$

$$x^{-\langle a\rangle} \triangleq (x^*)^{\langle a\rangle} = (x^{\langle a\rangle})^* \quad (9)$$

By using equation (7), the authors in [7] defined the Fractional Lower-order Wigner-Ville Distribution (FLOWVD) given by

$$W_x^{\langle a\rangle}(t,f) = \int_{-\infty}^{\infty} x^{\langle a\rangle}(t+\tau/2)\, x^{-\langle a\rangle}(t-\tau/2)\, e^{-j2\pi f\tau} d\tau \quad (10)$$

The fact that FLOWVD is a quadratic TFD, it inherits the same limitations of the WVD in the case of polynomial phase signals of order $p > 2$ as discussed in section 2. Figure 5 shows the FLOWVD for PPS of degree $p = 3$ in α-stable noise with $\alpha = 1$. We propose the Fractional Lower-Order Polynomial WVD (FLOP-WVD) to process polynomial phase signals of order $p > 2$ which

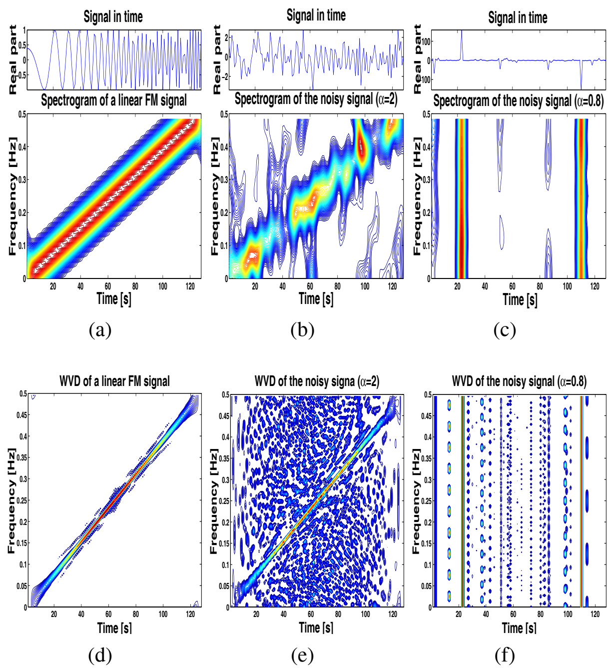

Fig. 2. The effect of α-stable additive noise with different distributions on the spectrogram and Wigner-Ville Distribution

is the generalization of the FLOWVD for non-stationary signals corrupted by alpha stable noise.
Let us, consider the noisy signal $x(t)$ defined in (6). We define the FLOPWVD as

$$W_x^{q,\langle a\rangle}(t,f) = \int_{-\infty}^{\infty} K_x^{q,\langle a\rangle}(t,\tau)e^{-j2\pi f\tau}d\tau \qquad (11)$$

where the kernel $K_x^g(t,\tau)$ is given by

$$K_x^{q,\langle a\rangle}(t,\tau) = \prod_{k=0}^{q/2}[x^{\langle a\rangle}(t+t_k\tau)]^{\gamma_k}[x^{-\langle a\rangle}(t+t_{-k}\tau)]^{-\gamma_{-k}} \qquad (12)$$

where $x^{\langle a\rangle}$ is defined in (8)-(9), and both parameters t_k, γ_k are obtained in the same way as for the standard PWVD.
The value of the parameter a has to taken such that $0 < a < \alpha/2$, the optimal value is an open field for research. In the following section, we will assess the performance of the FLOPWVD with respect to the standard PWVD and the FLOWVD for the case of a polynomial phase signal of order $p = 3$.

6. SIMULATION RESULTS

In this section, the performance gain when using fractional lower order statistics based PWVD is shown. In our simulation as mentioned in section 2, we used the sixth order PWVD kernel given in [11]

$$\begin{aligned} K_x^6(t,f) = & [x(t+0.62\tau)x^*(t-0.62\tau)] \\ & \times[x(t+0.75\tau)x^*(t-0.75\tau)] \\ & \times[x(t-0.87\tau)x^*(t+0.87\tau)]. \end{aligned} \qquad (13)$$

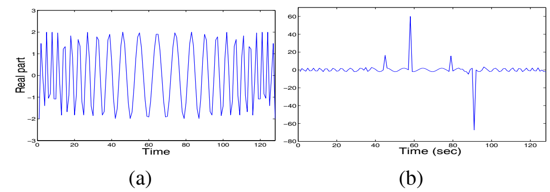

Fig. 3. Time history of (a): a PPS of order $p = 3$, (b): the same signal corrupted by impulsive noise

The used noiseless signal is a PPS of order $p = 3$ given by $z(t) = 2e^{j2\pi(a_0+a_1t+a_2t^2+a_3t^3)}$, where the signal's phase coefficients are given by $a_0 = 0, a_1 = 0.1, a_2 = 0$, and $a_3 = 2.44 \cdot 10^{-5}$.
We consider a symmetric alpha stable noise SαS ($\beta = 0$) with different values of the α parameter. The noisy signal is sampled at $T = 1$ and the number of observations N is chosen equal to 128. The time history of the noiseless and noisy signal respectively are shown in Figure 3.
The first step is to present the PWVD applied to an impulsive noise corrupted signal as shown in Figure 4 where one observes its limitation. Now, by using the Fractional Lower-Order Kernel as given below

$$\begin{aligned} K_x^{6,\langle a\rangle}(t,f) = & [x^{\langle a\rangle}(t+0.62\tau)x^{-\langle a\rangle}(t-0.62\tau)] \\ & \times[x^{\langle a\rangle}(t+0.75\tau)x^{-\langle a\rangle}(t-0.75\tau)] \\ & \times[x^{\langle a\rangle}(t-0.87\tau)x^{-\langle a\rangle}(t+0.87\tau)]. \end{aligned} \qquad (14)$$

We display in figure 6 the result when the new robust polynomial Wigner-Ville distribution is applied to the signal corrupted by Cauchy noise ($\alpha = 1$). Figure 7 shows the FLOPWVD in the case of a signal corrupted by symmetric α-stable noise ($\beta = 0$) with $\alpha = 0.5$. In both cases we took $a = 0.1$ and $\gamma = 0.2$. From figures 6 and 7, one can observe clearly that FLOPWVD performs better than the standard PWVD as well as the FLOWVD.

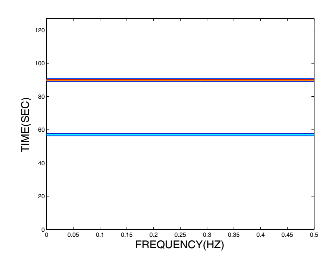

Fig. 4. PWVD of the noisy signal with $\alpha = 0.5$

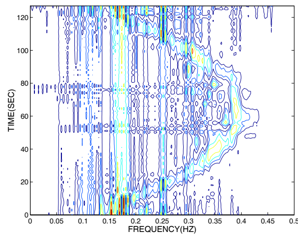

Fig. 5. The FLOWVD of the noisy signal with $\alpha = 1$

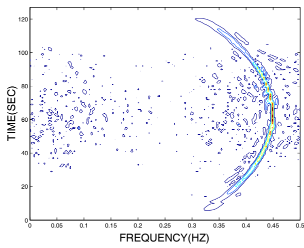

Fig. 6. FLOPWVD with $\alpha = 1$

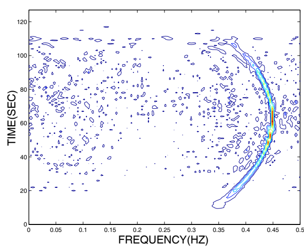

Fig. 7. FLOPWVD with $\alpha = 0.5$

7. CONCLUSION

In this paper, we have discussed the effect of additive heavy-tailed noise on the performance of different time-frequency distributions. In particular, we investigated the influence of impulsive noise modelled by α-stable distributions on the WVD and PWVD in the case of polynomial phase signals. The α-stable noise degrades badly the TFDs as shown in simulations. By taking advantage of Fractional Lower-Order Statistics that are known to be robust to impulsive noise robustness, we proposed as an extension to previous works a new Polynomial Wigner-Ville Distribution based on fractional lower order statistics (FLOPWVD) for the representation of the IF of a polynomial FM signal of order > 2 embedded in α-stable noise. From simulations, we observed that the approaches considered in this paper performed significantly better than the standard PWVD.

8. REFERENCES

[1] S. Barbarossa, A. Scaglione, and G. B. Giannakis, "Product high-order ambiguity function for multicomponent polynomial-phase signal modeling," *IEEE Trans. on Signal Processing*, vol. 46, pp. 691–708, March 1998.

[2] B. Boashash and P. O'Shea, "Polynomial wigner-ville distributions and their relationship to time-varying higher order spectra," *IEEE Transactions on Signal Processing*, vol. 42, Issue: 1, pp. 216–220, January 1994.

[3] B. Barkat, "If estimation of higher-order polynomial fm signals corrupted by multiplicative and additive noise," *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP '01).*, vol. 6, pp. 3561–3564, 2001.

[4] A. Ould-Ali and M. Benidir, "Statistical analysis of polynomial phase signals affected by multiplicative and additive noise," *Signal processing*, vol. 78, N1, pp. 19–42, 1999.

[5] I. Djurovic and L. Stankoic, "Influence of high noise on the instantaneous frequency estimation using quadratic time-frequency distributions," *IEEE Signal Processing Letters*, vol. 7, Issue: 11, pp. 317–319, Nov 2000.

[6] D. Middleton, "Channel modelling and threshold signal processing in under-water acoustics: An analytical overview," *IEEE J. Oceanic Eng.*, vol. OE-12, pp. 4–28, 1987.

[7] D. W. Griffith Jr, J. G. Gonzalez, and G. R. Arce, "Robust time-frequency representations for signals in stable noise using fractional lower-order statictics," *Proceedings of the IEEE Signal Processing Workshop on Higher-Order Statistics*, pp. 415–419, July 1997.

[8] M. Shao and C.L. Nikias, "Signal processing with fractional lower order moments: stable processes and their applications," *Proceedings of the IEEE*, vol. 81, Issue: 7, pp. 986 –1010, July 1993.

[9] B. Boashash, "Time-frequency signal analysis," *in Advances in Spectral Estimation and Array Processing (S. Haykin, Ed.)*, vol. 1, pp. 418–517, 1991.

[10] M. Benidir and A. Ouldali, "Polynomial phase signal analysis based on the polynomial derivatives decompositions," *Signal processing*, vol. 47, N7, pp. 1954–1956, 1999.

[11] B. Barkat and B. Boashash, "Design of higher order polynomial wigner-ville distributions," *IEEE Transactions on Signal Processing*, vol. 47, NO. 9, pp. 2608–2611, September 1999.

[12] D. Middleton, "Statistical physical models of electromagnetic interference," *IEEE Transactions on Electromagn. Compat.*, vol. EMC-19, pp. 106–127, August 1977.

[13] X. Ma and C.L. Nikias, "Joint estimation of time delay and frequency delay in impulsive noise using fractional lower order statistics," *IEEE Transactions on Signal Processing*, vol. 44 Issue: 11, pp. 2669 –2687, November 1996.

[14] G.A. Tsihrintzis, U. Tureli, and C.L. Nikias, "Fractional lower-order statistics-based ambiguity functions for differential delay doppler estimation," *IEE Proceedings Radar, Sonar and Navigation*, vol. 143 Issue: 6, pp. 358 –365, December 1996.