INTERPOLATION WITH NONUNIFORM B-SPLINES

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ABSTRACT

We consider the problem of interpolating a continuous-time signal from a set of *uniformly* spaced discrete-time samples. One of the prime methods of interpolation is based on the use of uniform B-splines. In this paper, we introduce a new interpolation approach using *nonuniform* B-splines as interpolation kernels. We show both theoretically, and through simulation results, that using nonuniform B-splines for interpolation of a signal from uniform samples can result in a higher quality of interpolation with respect to uniform B-splines, at the same computational cost.

1. INTRODUCTION

Interpolation is at the heart of digital signal and image processing theory, and arises in a variety of applications [1]. It is used to estimate intermediate values of a continuous-time (CT) signal $f(x)$ from its discrete-time (DT) samples $f_n = f(x_n)$. Examples in which interpolation is required include image transformations such as rotation, image rescaling (reduction or magnification), and data resampling. There are also applications where we can benefit from introduction of efficient interpolation algorithms, for example data compression and image warping.

The principal of interpolation is to represent a CT signal as a linear combination of shifts of some interpolation kernel. The Shannon-Whittaker theory provides a perfect reconstruction of bandlimited functions from uniformly spaced samples, when $\text{sinc}(x)$ is chosen as the interpolation kernel. Direct implementation of $\text{sinc}(x)$ is impossible due to its infinite support. Still, this method is widely used in the front end of digital systems, with various finite approximations of $\text{sinc}(x)$. Another interpolation approach is based on the design of interpolation kernels that are piecewise-polynomial with short support [2].

Traditionally, interpolation kernels were designed to satisfy the interpolation constraint, which requires the kernel to vanish at all sampling points except the origin. However, in recent years this constraint has been relaxed, and more general interpolation kernels have been designed, which do not necessarily satisfy the interpolation constraint [3], [4]. Generalized interpolation, which we review in Section 2, provides a wider choice for interpolation kernels, which may have better interpolation properties. To analyze the performance of different interpolation kernels, in Section 3, we introduce the Fourier error kernel as a basic tool for characterization of the approximation error for a given interpolation kernel [5]. The most popular generalized interpolation functions, due to their excellent interpolation properties and short support, are uniform B-splines [3], or linear combinations of uniform B-splines and their derivatives [4]. Numerous experiments [1] have shown their advantages over traditional interpolation kernels.

In this paper, we introduce a new class of generalized interpolation kernels, which lead to efficient interpolation in the squared-norm sense. Our approach is based on using nonuniform B-splines, which we define in Section 4, as interpolation kernels. In Section 5 we show that using nonuniform B-splines can reduce the interpolation error, with no additional computational cost.

2. GENERALIZED INTERPOLATION

We consider the problem of interpolating a CT signal $f(x)$, from its uniform samples $f_n = f(n)$, where we assume for simplicity that the sampling period T is equal to 1. The generalized interpolation approach interpolates a CT signal $f_{int}(x)$ as a linear combination of uniform shifts of an interpolation kernel $\varphi(x)$

$$f_{int}(x) = \sum_{k \in Z} c_k \varphi(x - k), \tag{1}$$

where c_n is a set of discrete values, that is uniquely determined by the samples f_n. In this paper, we restrict our attention to consistent interpolation methods, *i.e.,* we require that $f_{int}(n) = f_n$.

Traditionally it was common to choose $c_n = f_n$. In this case, to guarantee consistent interpolation, the interpolation kernel must satisfy the interpolation condition $\varphi(n) = \delta[n]$, where $\delta[n]$ is the Kronecker delta function. A classical example is the $\text{sinc}(x)$ interpolation kernel.

For more general interpolation kernels, the interpolation (1) is consistent if

$$f_n = f_i(n) = \sum_{k \in Z} c_k \varphi(n - k) = (c * b)_n, \tag{2}$$

where $b_n = \varphi(n)$ and $(c*b)_n$ denotes the DT convolution between c_n and b_n, which is equivalent to digital filtering. From (2) if follows that the coefficients c_n can be found by inverse convolution as

$$c_n = ((b)^{-1} * f)_n, \tag{3}$$

where $(b)^{-1}$ is the convolution-inverse of b, and its z-transform is given by

$$B^{-1}(z) = \frac{1}{\sum_{k \in Z} b_k z^k}. \tag{4}$$

Assuming the interpolation kernel $\varphi(x)$ is symmetric, the filter (4) can be implemented efficiently by decomposing it into a causal and an anti-causal filter; for details see [6].

The most popular generalized interpolation kernels are uniform B-splines [3]. In Section 5 we show that the quality of the interpolation can be improved by using nonuniform B-splines as interpolation kernels, in place of uniform B-splines. To this end, in the next section we consider mathematical tools for analyzing the performance of different interpolation kernels.

3. ERROR ANALYSIS

3.1. Error Kernel

It was shown in [5] that if $f(x)$ is bandlimited to $[-\pi, \pi]$, then

$$\eta^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 E_{int}(\omega) d\omega, \quad (5)$$

where $\eta = \|f(x) - f_{int}(x)\|_{L_2}$ is the squared-norm of the interpolation error, and $E_{int}(\omega)$ is an interpolation *error kernel* that depends on the interpolation kernel $\varphi(x)$ only, and is given by

$$E_{int}(\omega) = 1 - \frac{\hat{\varphi}(\omega)^2}{\hat{a}(\omega)} + \left|\frac{\sqrt{\hat{a}(\omega)}}{\hat{b}(\omega)} - \frac{\hat{\varphi}(\omega)}{\sqrt{\hat{a}(\omega)}}\right|^2. \quad (6)$$

Here $\hat{f}(\omega)$, $\hat{\varphi}(\omega)$, $\hat{b}(\omega)$, and $\hat{a}(\omega)$ are the Fourier transforms of $f(x)$, $\varphi(x)$, $\varphi(k)$, and $\varphi(k) * \varphi(-k)$, respectively.

If $f(x)$ is not bandlimited, then η is an average measure of the error over all possible shifts of the sampling set, *i.e.,* $f_n = f(n + \tau)$, where $\tau \in [0, 1]$. For both bandlimited and nonbandlimited cases, η provides a good characterization of the interpolation error.

3.2. Order of Approximation

When the sampling step T is small, the behavior of the interpolation error can be characterized by the *approximation order* L of the interpolation kernel $\varphi(x)$. An interpolation kernel has approximation order L if [7]:

$$\begin{array}{ll} \hat{\varphi}(2\pi k) = \delta[k], & k \in Z; \\ \hat{\varphi}^{(N)}(2\pi k) = 0, & k \in Z, \quad N = 1, \ldots, L-1, \end{array} \quad (7)$$

where $\hat{\varphi}^{(N)}(\omega)$ is the Nth derivative of the Fourier transform of $\varphi(x)$. For such a kernel, the interpolation error decreases like T^L [8], *i.e.,*

$$\|f(x) - f_{int}^T(x)\| \le C_L T^L \|f^{(L)}(x)\| \quad \text{as } T \to 0, \quad (8)$$

where C_L is a known constant, $f^{(L)}(x)$ is the Lth derivative of $f(x)$, and $f_{int}^T(x)$ is an interpolated version of $f(x)$ from the samples $f(nT)$, obtained by (1).

Uniform B-splines have order $L = n + 1$, which is the maximal for piecewise-polynomials of degree n. Although, the approximation order of nonuniform B-splines, which we define in the next section, is zero, they can provide good results for $T \gg 0$.

4. NONUNIFORM B-SPLINES

A B-spline $N^n(x)$ of degree n is a piecewise polynomial with $n+1$ pieces that are smoothly connected together. The joining points of the polynomials are called *knots*. For a B-spline of degree n, each segment is polynomial of degree n. The special case of B-splines with uniform knots was studied extensively by Unser *et al.* [3]. Here, we will consider B-splines with nonuniform knots, which are defined as [9]

$$N^n(x) = (x_{n+1} - x_0) \sum_{i=0}^{n+1} \frac{(x_i - x)_+^n}{\prod_{l=0, l\neq i}^{n+1}(x - x_l)}, \quad (9)$$

where $\{x_i\}_{i=0}^{n+1}$ are the knots with $x_i < x_{i+1}$ and

$$(x_i - x)_+^n = \left\{ \begin{array}{ll} (x_i - x)^n, & x_i \geqslant x; \\ 0, & x_i < x. \end{array} \right. \quad (10)$$

We see from (9) that $N^n(x) > 0$, and $N^n(x) = 0$ outside the region $[x_0, x_{n+1}]$. We define the support of $N^n(x)$ as $W = x_{n+1} - x_0$. As two special cases of (9), the B-spline of degree 0 is given by

$$N^0(x) = \left\{ \begin{array}{ll} 1, & x_0 \leqslant x < x_1; \\ 0, & \text{otherwise,} \end{array} \right. \quad (11)$$

and the B-spline of degree 1 is given by

$$N^1(x) = \left\{ \begin{array}{ll} \frac{x - x_0}{x_1 - x_0}, & x_0 \leqslant x < x_1; \\ \frac{x_2 - x}{x_2 - x_1}, & x_1 \leqslant x < x_2; \\ 0, & \text{otherwise.} \end{array} \right. \quad (12)$$

These two functions correspond to zero-order-hold and linear interpolation functions, and both satisfy the basic interpolation property. Examples of B-splines of degrees 0 to 3 with knots chosen arbitrarily are shown in Fig. 1.

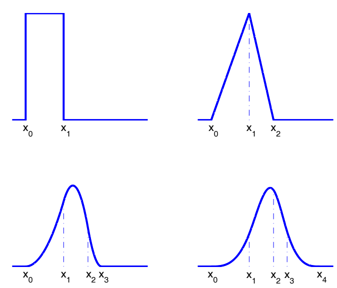

Fig. 1. Nonuniform B-splines of degree 0 to 3.

An important property of the B-spline in the context of interpolation, is that it's derivatives are continuous up to order $(n-1)$ at the knots. We can also see in Fig. 1 that B-splines of degree greater than one are very smooth functions, which makes them attractive as interpolation kernels. The B-spline can be computed recursively as [10]

$$N^n(x) = \frac{x - x_0}{x_n - x_0} N_0^{n-1}(x) + \frac{x_{n+1} - x}{x_{n+1} - x_1} N_1^{n-1}(x), \quad (13)$$

where $N_0^{n-1}(x)$ and $N_1^{n-1}(x)$ are B-splines of degree $n-1$ with knots $\{x_i\}_{i=0}^{n}$ and $\{x_i\}_{i=1}^{n+1}$ respectively. Applying (13) n times, a B-spline of degree n consists of $n+1$ polynomial pieces, each of degree n:

$$N^n(x) = \sum_{i=0}^{n}\sum_{k=0}^{n} c_{ik} x^k N_i^0(x), \quad (14)$$

where $N_i^0(x)$ is a B-spline of degree 0 with knots x_i, x_{i+1}. The Fourier transform of $N_i^0(x)$ is

$$\hat{N}_i^0(\omega) = e^{-j\omega s_i} 2a_i \text{sinc}(a_i w), \quad (15)$$

where $a_i = (x_{i+1} - x_i)/2$ and $s_i = (x_{i+1} + x_i)/2$. Using the fact that

$$x^k f(x) \overset{\text{Fourier}}{\longleftrightarrow} j^k \frac{d^k}{d\omega^k}\hat{f}(\omega), \quad (16)$$

together with (14) and (15), we conclude that the Fourier transform of a nonuniform B-spline of degree n is

$$\hat{N}^n(\omega) = \sum_{i=0}^{n}\sum_{k=0}^{n} 2a_i c_{ik} j^k \frac{d^k}{d\omega^k}\{e^{-j\omega s_i}\text{sinc}(a_i w)\}. \quad (17)$$

To prevent phase degradation, it is desirable to have a symmetric interpolation kernel. To guarantee that the B-spline $N^n(x)$ is symmetric, in what follows we choose the knots $\{x_i\}_{i=0}^{n+1}$ to be symmetric, *i.e.,* $x_i = -x_{n+1-i}$.

Another important property of the interpolation kernel is its support W. Most of the piecewise-polynomial interpolation kernels, among them uniform B-splines, have support $W = n + 1$, where n is their degree [1]. To compare performance of nonuniform B-splines to other interpolation kernels we fix its support to $W = n + 1$. This choice automatically places the first (x_0) and the last (x_{n+1}) knots at $-W/2$ and $W/2$, respectively.

Examples of symmetric B-spline functions of degree 3 are presented in Fig. 2.

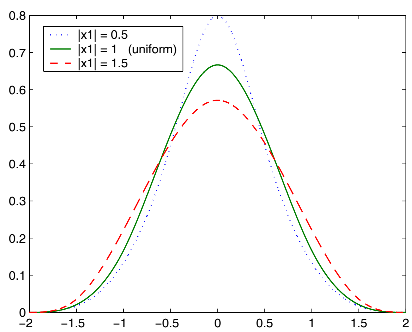

Fig. 2. Cubic B-splines of support $W = 4$ with $|x_1| = 0.5$, $|x_1| = 1$ (corresponding to a uniform B-spline), $|x_1| = 1.5$, and $x_3 = -x_1$.

5. INTERPOLATION WITH NONUNIFORM B-SPLINES

In this section, we analyze the use of a nonuniform B-spline $N^n(x)$, defined by (9), as an interpolation kernel. We compare the performance of nonuniform B-splines with given degree n and support W, with uniform B-splines.

5.1. Error Kernel Analysis

To evaluate the interpolation error, we use the error kernel (6). $E_{int}(\omega)$ of several B-splines functions of degree 3 are shown in Fig. 3. We can see that below the Nyquist frequency $w = \pi$, the kernel is the smallest for $|x_1| = 1.5$. This suggests that the squared-norm error for interpolation of bandlimited functions will be smaller when using a B-spline with $|x_1| = 1.5$ than for the other presented in the figure, including the uniform B-spline. When interpolating a signal that is not bandlimited, *i.e.,* its spectrum includes high frequency components (above $w = \pi$), a B-spline with $|x_1| = 0.5$ can provide better interpolation quality.

To choose the B-spline interpolation kernel, we first calculate the interpolation error η for different B-splines, with certain spectrum models $\hat{f}(\omega)$. Then, among B-splines with the same degree n and support W, we choose the one with optimal knot placements, that results in the highest SNR value, where SNR $= 10\log_{10}(1/\eta^2)$. The analysis for different signal models is presented in Sections 5.3 and 5.4.

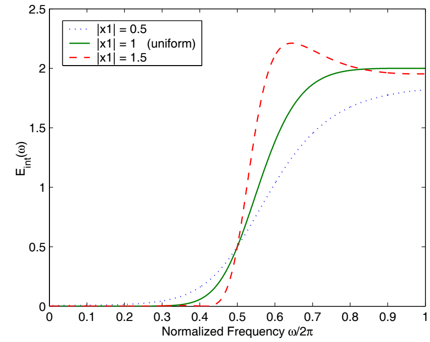

Fig. 3. $E_{int}(\omega)$ for cubic B-splines with $|x_1| = 0.5$, 1, and 1.5.

5.2. Approximation Order Analysis

Using (7), we can show that the approximation order of nonuniform B-splines is zero, except for a few special cases among them uniform B-splines, whose approximation order is $L = n+1$. As a result, the uniform B-splines will perform better than nonuniform B-splines for small values of T.

There are many practical applications where data can not be sampled at a given rate. Furthermore, when dealing with q-dimensional signals, to save computations and storage space which increase exponentially like $(1/T)^q$, we are typically interested in large values of T. In such cases, the approximation order L is not a good criterion for error characterization.

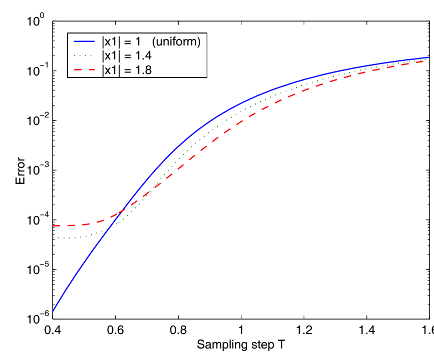

Fig. 4. Error curves for the interpolation of $f(x) = -xe^{-x^2}$ with cubic B-splines for $|x_1| = 1$, 1.4, and 1.8.

The behavior of the error as a function of the sampling step T is shown in Fig. 4, where a scaled first derivative of a Gaussian was used as a test function. The squared error was obtained by numerical integration after interpolating with a B-spline. The solid curve corresponds to the interpolation with a uniform cubic B-spline. As predicted by (8), in this case the interpolation error decreases like T^4 as the sampling step tends to zero. Since the order of approximation of two other B-splines used in the example is zero, their interpolation error decreases slowly for small values of T. However, when the sampling step T grows, the interpolation error is smaller when using the nonuniform B-splines. The B-spline with $|x_1| = 1.8$ achieves 7.4 dB improvement over the uniform B-spline, when the sampling step is $T = 1$.

5.3. Interpolation of Bandlimited Signals

To comply with Nyquist sampling theory we assume that the original CT signal $f(x)$ is bandlimited to $[-\pi, \pi]$. We also assume that it has a constant-value power spectrum, when for other spectrum models we obtained similar results. In this case, the squared-norm interpolation error is obtained by integrating the error kernel itself (5). The SNR results of these integrations for different nonuniform B-splines as a function of the knot placements are presented in Fig. 5. The top two figures corresponds to symmetric B-splines of degree 2 (left) and 3 (right). B-splines of degree $4-5$ (bottom) have two knots (x_1 and x_2) which can be changed under the restriction that $|x_1| > |x_2|$. In these two bottom figures lighter tone corresponds to higher SNR.

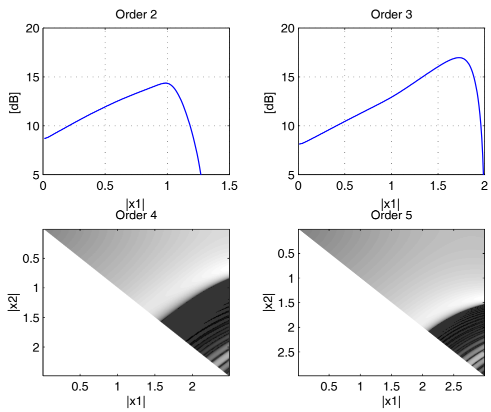

Fig. 5. Performance of symmetric B-splines of degree 2 (top-left) to 5 (bottom-right) as function of the knot placement, when interpolating a bandlimited signal with constant spectrum.

In Table 1, we summarize the results for nonuniform symmetric B-splines of degrees 2 to 7. The knot placement maximizing the SNR is given in the table together with the optimal SNR value. We also present the SNR values for interpolation with uniform B-splines. For cubic B-splines, which are widely used in practice, there is an improvement of 4 dB over the uniform B-spline.

Table 1. Optimal Knot Placement of Nonuniform B-Splines for Interpolation of Bandlimited Signals with Constant Spectrum

B-Spline Degree	Knot Placement $\lvert x_0\rvert$	$\lvert x_1\rvert$	$\lvert x_2\rvert$	$\lvert x_3\rvert$	SNR [dB] Optimal	Uniform
2	1.5	0.99			14.47	12.12
3	2.0	1.73			17.17	13.15
4	2.5	2.49	0.67		19.50	14.18
5	3.0	2.99	1.41		20.19	14.94
6	3.5	3.49	2.54	0.06	23.31	15.62
7	4.0	3.97	3.29	1.21	24.39	16.19

Simulation results, where a CT signal $f(x)$ is randomly generated and sampled at the Nyquist rate, show an improvement in interpolation error when interpolating with the optimal B-splines, which are in general nonuniform.

5.4. Interpolation of Images

In the multidimensional case, to reduce the computational load, we assume that the interpolation kernel is separable.

A common model to describe the transform of images is the Markov model, in which the image $f(x)$ is assumed to satisfy $\int_{-\infty}^{\infty} f(x)f(x+\tau)dx = \rho^{-|\tau|}$, with $\rho = 0.9$ [11]. In this case

$$|\hat{f}(\omega)|^2 = \frac{-2\ln(\rho)}{\omega^2 + (\ln(\rho))^2}. \tag{18}$$

Substituting (18) into (5) and integrating the error for different B-splines of degree 3, we found that $|x_1| = 0.78$ is the optimal knot placement and its SNR is 14.82 dB, which is just 0.1 dB more than the SNR of a uniform B-spline.

In order to obtain a quantitative comparison of different cubic B-splines, we magnify an image by a factor of 2. Since the result should be compared to the original image, the magnification operation was applied to a subsampled version of the original image. For images whose spectrum include high frequencies, *i.e.,* correspond to the Markov model (18), the average improvement achieved with the optimal B-spline over the uniform B-spline is 0.1 dB (which complies with the theory). For smooth images, with spectrum concentrated mostly around $\omega = 0$, the uniform B-spline provides better results due to its high approximation order.

6. REFERENCES

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