GENERALIZED COMB FUNCTION: A NEW SELF-FOURIER FUNCTION

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ABSTRACT

The comb function defined as equidistantly spaced impulses (i.e., an impulse train) is well known that its Fourier transform provides also a comb function, and is used for a proof of the sampling theorem. As a generalization of this function, we propose a novel comb function, called "generalized comb function (GCF)", that consists of equally spaced but proportionally expanded pulses along the trans-versal axis. It is shown that the Fourier transform of the GCF with an arbitrary pulse shape can be obtained only by replacement of variables without any Fourier integral operation, and the transformed function is also included in the GCF family, like that of the conventional comb function. The theorem representing this relationship and some examples are presented.

1. INTRODUCTION

The comb function defined as an infinite series of equidistantly spaced Dirac's delta functions is a useful mathematical tool for various signal analyses such as a proof of the sampling theorem, a mathematical representation of periodic signals and so on. The most important characteristic is that the Fourier transform of the comb function provides also a comb function with the reciprocal spacing of the original interval. For example, the comb filter [1] is well known as an ideal filter for separating a periodic signal from the noise-corrupted observation, of which the impulse response has a comb function with the pulse interval equal to the pitch period. Therefore the transfer function takes also a comb function with the line spectrum spacing corresponding to the fundamental frequency.

As a generalization of the comb function, we propose a novel comb function, called "generalized comb function (GCF)", that consists of equally spaced but proportionally expanded pulses along the transverse axis. In previous works [2, 3], the prototypes of this function have been derived as an impulse response or a transfer function of the optimal filters for extracting a quasi-periodic signal involving random pitch fluctuations, but no relationship between the Fourier transform pair has been clarified. In this paper, we show that the GCF has a remarkable feature such that the Fourier transformed function is also included in the GCF family, like that of the conventional comb function.

Such a function that holds an unchangeability to the Fourier transform as the conventional or our proposed comb functions is called "self-Fourier function (SFF)." Another examples of SFF have also been found, e.g., the Gaussian, Hermite-Gaussian functions, $1/\sqrt{|t|}$ and $\mathrm{sech}(\pi t)$ [4, 5, 6]. While these examples have definite expressions, Caola [7] has showed that various SFF's can be constructed by summation of an arbitrary function and its Fourier transformed function. The reason such functions hold the SFF property is that each term of that function is just interconverted through the Fourier transform. Our proposed SFF, i.e., the GCF, can also be constructed from any arbitrary function, but is based on the quite different concept from the Caola's function. The notable feature is that the GCF can be represented as a weighted superposition of various comb functions with different spacing. From the SFF property of each comb function, it is shown that the Fourier transform of that function can be obtained only by replacement of variables without any Fourier integral operation that is necessary for constructing the Caola's function.

2. GENERALIZED COMB FUNCTION AND ITS FOURIER TRANSFORM

Definition: The *generalized comb function* (GCF) $h(t)$ is defined as

$$h(t) = A\delta(t) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{|n|} h_0(\frac{t}{n}), \tag{1}$$

where

$$A = \int_{t_a}^{t_b} h_0(t)dt$$

and $h_0(t)$ is an arbitrary function with a finite support $[t_a, t_b]$ and piecewise smoothness. In the special case of $h_0(t) \rightarrow A\delta(t)$, this function just falls into the well-known conventional comb function.

Then it satisfies the following relation.

Theorem: The Fourier transform of a GCF

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt \tag{2}$$

can be represented as

$$H(f) = B\delta(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{|f|} h_0(\frac{n}{f}) \tag{3}$$

and is also included in the GCF family defined by (1), where

$$B = \int_{t_a}^{t_b} \frac{h_0(t)}{t} dt.$$

Proof: (1) can be rewritten as

$$h(t) = \int_{t_a}^{t_b} h_0(T) \sum_{n=-\infty}^{\infty} \delta(t - nT) dT \tag{4}$$

because the right side

$$\begin{aligned}
&= \int_{t_a}^{t_b} h_0(T) dT \cdot \delta(t) + \int_{t_a}^{t_b} h_0(T) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \delta(t - nT) dT \\
&= A\delta(t) + \int_{t_a}^{t_b} h_0(T) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{|n|} \delta(T - \frac{t}{n}) dT \\
&= A\delta(t) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \int_{t_a}^{t_b} h_0(T) \frac{1}{|n|} \delta(T - \frac{t}{n}) dT \\
&= \text{the left side of (1)},
\end{aligned}$$

where the first column to the second is derived by a variable-substitution's rule of delta function, i.e., the formulation $\delta(c(T)) = \frac{1}{|c'(T_0)|}\delta(T - T_0)$ where $c(T) = t - nT$ and T_0 is t satisfying $c(T_0) = 0$. Using the above (4), the Fourier transform can be calculated as follows:

$$\begin{aligned}
H(f) &= \int_{t_a}^{t_b} h_0(T) \mathcal{F}\{\sum_{n=-\infty}^{\infty} \delta(t - nT)\} dT \\
&= \int_{t_a}^{t_b} h_0(T) \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T}) dT \\
&= \int_{t_a}^{t_b} h_0(T) \frac{1}{T} dT \cdot \delta(f) \\
&\quad + \int_{t_a}^{t_b} h_0(T) \frac{1}{T} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \delta(f - \frac{n}{T}) dT,
\end{aligned} \tag{5}$$

where $\mathcal{F}\{\cdot\}$ means a Fourier transform operator and the first column to the second is given by a well-known Fourier transform rule of comb function. Using a variable-substitution's rule of delta function with $c(T) = f - \frac{n}{T}$ again, the above is equal to (3) as follows:

$$\begin{aligned}
H(f) &= B\delta(f) + \int_{t_a}^{t_b} h_0(T) \frac{1}{T} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{|n|}{f^2} \delta(T - \frac{n}{f}) dT \\
&= B\delta(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \int_{t_a}^{t_b} h_0(T) \frac{|n|}{T f^2} \delta(T - \frac{n}{f}) dT \\
&= \text{the right side of (3)}.
\end{aligned} \tag{6}$$

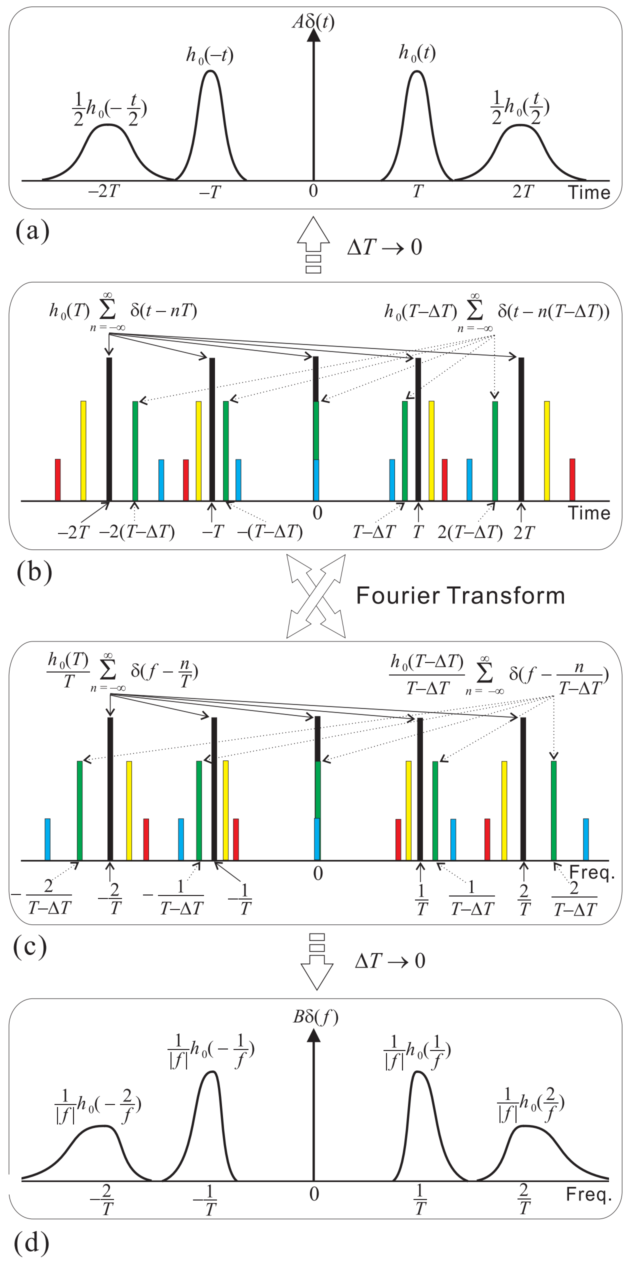

Fig. 1. Fourier transform of the generalized comb function.

Assigning the kernel function in (1) to the following

$$H_0(f) = \frac{1}{|f|} h_0(\frac{1}{f}), \tag{7}$$

(3) can be rewritten as

$$H(f) = B\delta(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{|n|} H_0(\frac{f}{n}) \tag{8}$$

where $H_0(f)$ has a finite support in $[\frac{1}{t_b}, \frac{1}{t_a}]$ and the following relation holds:

$$\int_{\frac{1}{t_b}}^{\frac{1}{t_a}} H_0(f) df = \int_{t_a}^{t_b} \frac{h_0(t)}{t} dt = B.$$

Consequently, (8) also satisfies the definition of the GCF by (1). ■

Fig.1 shows a schematic diagram of the above proof. It illustrates that (4) consists of a weighted superposition of various comb functions with different spacing (see Fig.1 (b)), in which each comb function is Fourier-transformed to another comb function with the reciprocal spacing (see (5) and Fig.1 (c)) and reconstructed to another GCF (see (6) and Fig.1 (d)). It should be noticed that the Fourier transform of a GCF could be obtained only by replacement of variables without any Fourier integral operation.

3. EXAMPLES

For verification of the relationship between (1) and (3), the GCF's with the following kernel functions are examined.

Example 1: A Gaussian function truncated at both ends

$$h_0(t) = \begin{cases} e^{-\frac{(t-T_0)^2}{2\sigma_T^2}}, & \text{for } T_0 - T_g < t < T_0 + T_g \\ 0, & \text{other.} \end{cases} \tag{9}$$

Example 2: A rectangular function

$$h_0(t) = \begin{cases} 1, & \text{for } T_0 - T_r < t < T_0 + T_r \\ 0, & \text{other.} \end{cases} \tag{10}$$

Each distribution constructed by (1) is shown in Figs.2(a) and 3(a) respectively, where the comb interval: $T_0 = 1$, the standard deviation: $\sigma_T = 0.01$, the truncation width: $T_g = 0.03$ and the rectangular width: $T_r = 0.02$ are chosen as an example. Figs.2(b) and 3(b) show the Fourier transforms obtained theoretically by substitution of (9) and (10) into (3), respectively.

For comparison with the theoretical results, distributions computed numerically by the discrete Fourier transform are shown in Figs.2(c) and 3(c), respectively. 256 discrete points per each interval between teeth of the comb, 128 teeth symmetrically on both the positive and negative axes, i.e., total 65536 points in time and frequency axes, are used for the computation. Only 16 intervals on the positive axis is, however, displayed in Figs.2 and 3. The numerical result in Fig.2(c) causes an error no larger than 0.01% compared with Fig.2(b) and apparently coincides with it. Also in the case of Fig.3, the numerical result (c) almost coincides with the theoretical one (b) except for the Gibbs vibrations appearing in discontinuous boundaries.

4. CONCLUSION

We proposed a new self-Fourier function (SFF), i.e., called "generalized comb function (GCF)" and proved that the Fourier transform takes also a GCF. The GCF can be constructed from an arbitrary kernel function with a finite support and piecewise smoothness, and the Fourier transform can be obtained only by inversion of the transverse axis without any Fourier integral operation. It implies that the GCF theory provides a new viewpoint about the SFF.

5. REFERENCES

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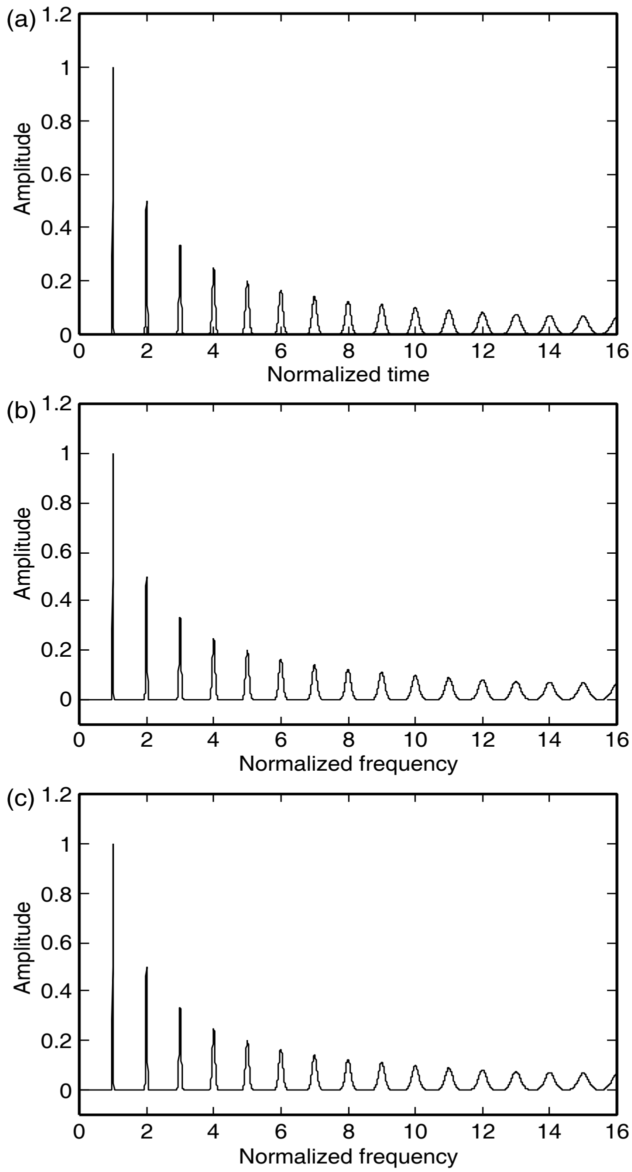

Fig. 2. A generalized comb function with 'Gaussian kernel' and its Fourier transform. (a) Original distribution in time domain. (b) Theoretical Fourier transform by substitution into (3). (c) Numerical computation by the discrete Fourier transform.

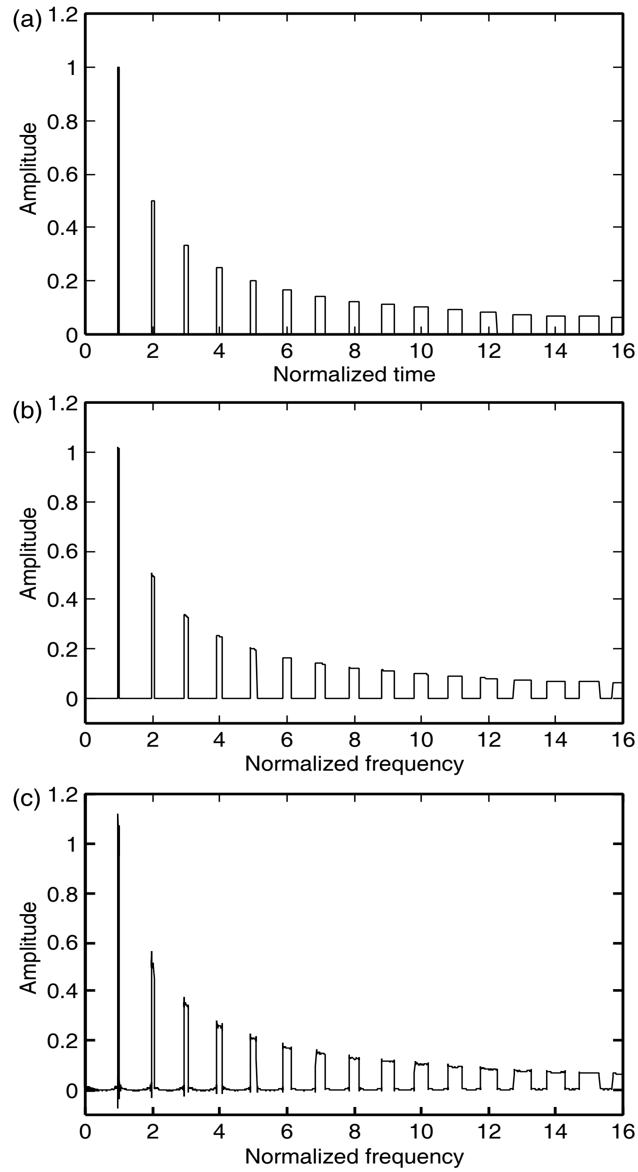

Fig. 3. A generalized comb function with 'rectangular kernel' and its Fourier transform. (a) Original distribution in time domain. (b) Theoretical Fourier transform by substitution into (3). (c) Numerical computation by the discrete Fourier transform.