CAN TIMING JITTER IMPROVE RANDOM PROCESS RECONSTRUCTION IN PRESENCE OF ALIASING ?

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ABSTRACT

In the case of uniform sampling, when the Nyquist condition is fulfilled, exact reconstruction of the time-continuous process can be achieved based on the observation of the samples. When the Nyquist condition is not fulfilled, the linear minimum mean square estimator (LMMSE) of the time-continuous process can be derived. In this paper, it is shown that in some cases, surprisingly, unobserved timing jitter can improve the quality of the LMMSE.

1. INTRODUCTION

The problem addressed in this paper is the study of the optimal linear reconstruction of a time-continuous process in the presence of both aliasing and jitter. Some papers and studies have pointed out the interest of timing jitter in the presence of aliasing effects. In [1], the effect of sampling jitter on the reconstructed spectrum of digitized signals is studied and additive random sampling's susceptibility to sampling jitter is analyzed at frequencies significantly above the Nyquist frequency limit. Kolsko [2] studies a three-dimensional spatio-temporal jittered sampling and shows that this model can be used to reduce aliasing due to undersampled frequency components. More recently, the EURODASP project (http://www.eurodasp.com/eurodasp) proposes to use a nonuniform sampling to achieve the alias suppression capability with subsequent digital signal processing. The question under the present paper is to understand whether the presence of a timing-jitter could help to reduce the aliasing effect.

In what follows, $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ is a (wide-sense) stationary zero-mean process with spectral density $s_Z(\omega)$ defined by

$$K_Z(\tau) = E[Z(t) Z^*(t-\tau)] = \int_{-\infty}^{+\infty} e^{i\omega\tau} s_Z(\omega) d\omega \tag{1}$$

where $E[-]$ holds for the mathematical expectation and * for the complex conjugate.

The periodic sampling of this random process leads to the sequence of observations $\mathbf{Z}_s = \{Z(n), n \in \mathbb{Z}\}$. For sake of simplicity and without any loss of generality, the sampling period is assumed to be equal to 1 second. The aim of this paper is to study the influence of both aliasing and jitter effects on the reconstruction of the continuous-time process based on the observed values of the samples.

The well-known sampling theorem states that a continuous-time band-limited process can be recovered from a set of samples in the case of periodic sampling if the sampling frequency is twice the spectral band limit of the process. Indeed, when the sampling theorem is verified, the reconstruction of the original continuous-time process can be achieved with a zero mean square error (MSE) by a linear interpolation of the samples. This interpolator is referred to as the Shannon interpolator [3]. When the hypotheses of the sampling theorem are not fulfilled, the linear reconstruction with a zero MSE is not possible. However, the linear minimum mean square estimator (LMMSE) $\hat{Z}(t)$ can be derived, with time-varying coefficients $b_n(t)$ function of $s_Z(\omega)$ [4], [5]:

$$\hat{Z}(t) = \sum_{n\in\mathbb{Z}} b_n(t) Z(n) \tag{2}$$

where
$$b_n(t) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mu_t(\omega) e^{-i\omega t} d\omega \tag{3}$$

and
$$\mu_t(\omega) = \frac{\sum_{k\in\mathbb{Z}} s_Z(\omega+2\pi k) e^{i2\pi kt}}{\sum_{k\in\mathbb{Z}} s_Z(\omega+2\pi k)} e^{i\omega t} \; for \; \omega \in [-\pi, +\pi]. \tag{4}$$

In this case, the reconstruction error power is

$$\begin{aligned} \sigma_t^2 &= E\left[\left|Z(t) - \hat{Z}(t)\right|^2\right] \qquad (5) \\ &= K_Z(0) - \int_{-\pi}^{+\pi} |\mu_t(\omega)|^2 \sum_{k\in\mathbb{Z}} s_Z(\omega+2\pi k) d\omega. \end{aligned}$$

2. JITTER AND ALIASING EFFECTS

However, the above results have been obtained under uniform sampling assumption. The problem of jitter in sampling has received an increasing attention in recent years because as sampling rates become higher, the effect of jitter on system performance can no longer be neglected. When sampling errors occur during the sampling process, the sampled observations can be written

$$U_n = Z\left(n - A_n\right) \tag{6}$$

where timing jitter is modeled by a random sequence $\mathbf{A}_s = \{A_n, n \in \mathbb{Z}\}$ independent of $\mathbf{Z}$. The jitter is assumed to be stationary in the sense that the two following characteristic functions [6] are independent of n

$$\begin{aligned} \Psi(\omega) &= E\left[e^{i\omega A_n}\right] \\ \Phi(m,\omega) &= E\left[e^{i\omega(A_n - A_{n-m})}\right]. \end{aligned} \tag{7}$$

The problem of finding the LMMSE of $Z(t)$ from the observations $\mathbf{U}_s = \{U_n, n \in \mathbb{Z}\}$ has been first addressed by Balakrishnan in [7]. Related studies can also be found for example in [8] and [9]. Note that the jitter sequence $\mathbf{A}_s$ is not observed and is only known by its characteristic functions given in (7). When the Nyquist condition is verified, the jitter introduces an estimation error which can be calculated as a function of $s_Z(\omega)$ and of the above characteristic functions (7).
Moreover, when both aliasing and jitter effects are present, the LMMSE of $Z(t)$ can also be derived. If the jitter random variables A_n are assumed to be independent, identically distributed (i.i.d.), the LMMSE is shown in the Appendix to be of the same form than in (2) except that the function $\mu_t(\omega)$ used in (3) takes into account the jitter statistics:

$$\begin{aligned} \mu_t(\omega) &= \frac{\sum_{k\in\mathbb{Z}} s_Z(\omega+2\pi k)\,\Psi(\omega+2\pi k)\,e^{i2\pi kt}}{\sum_{k\in\mathbb{Z}} s_Z(\omega+2\pi k)\left|\Psi(\omega+2\pi k)\right|^2 + c} e^{i\omega t} \\ for \quad \omega &\in [-\pi,+\pi] \end{aligned} \tag{8}$$

where c is a constant given by

$$c = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(1 - \left|\Psi(\omega)\right|^2\right) s_Z(\omega)\,d\omega. \tag{9}$$

In this case, the reconstruction error power σ_t^2 is of the same form than in (5). The only difference lies in the function $\mu_t(\omega)$, given by (4) for the case without jitter and by (8) for the case with jitter. The main idea of the present paper is to investigate whether in some cases, jitter could help to decrease the reconstruction error power when the Nyquist condition is not fulfilled. In fact, the power spectral density of the sampled process in presence of both aliasing and jitter effects is (see Annex):

$$\begin{aligned} s_{U_s}(\omega) &= \sum_{k\in\mathbb{Z}} s_Z(\omega+2\pi k)\left|\Psi(\omega+2\pi k)\right|^2 + c \\ for\ \omega &\in [-\pi,+\pi]. \end{aligned} \tag{10}$$

The aliasing effect is highlighted by the presence of the sum in (10), while the jitter effect yields the term function of $\Psi(\omega)$ (jitter statistics). Therefore, jitter can be seen as a bandpass filter which may weaken the frequency band where aliasing is the more important, thus allowing to fight against aliasing. But this benefit is compensated by the addition of a noise whose power c is directly linked to the presence of the jitter (for the jitter-free case, $c = 0$).
When the Nyquist condition is not fulfilled, in order to compare the reconstruction of the time-continuous process with and without jitter, we propose to define a measure of error independent of time instant t. Therefore, considering that the coefficients $b_n(t)$ defined in (3) have a unitary time instant period, the quantity ε^2 is defined by

$$\varepsilon^2 = \int_0^1 \sigma_t^2\,dt. \tag{11}$$

In what follows, this mean error will be denoted by ε_0^2 for the non-jitter case and ε^2 for the case where jitter is present. The problem is to study the sign of $\varepsilon^2 - \varepsilon_0^2$ as a function of the jitter statistics and of the spectral density $s_Z(\omega)$. In case of i.i.d. jitter random variables, this difference is of the form:

$$\begin{aligned} \varepsilon^2 - \varepsilon_0^2 &= \int_{-\pi}^{+\pi} \frac{\sum_{k\in\mathbb{Z}} s_Z^2(\omega+2\pi k)}{\sum_{k\in\mathbb{Z}} s_Z(\omega+2\pi k)} \\ &\quad - \frac{\sum_{k\in\mathbb{Z}} s_Z^2(\omega+2\pi k)\left|\Psi(\omega+2\pi k)\right|^2}{\sum_{k\in\mathbb{Z}} s_Z(\omega+2\pi k)\left|\Psi(\omega+2\pi k)\right|^2 + c} d\omega. \end{aligned} \tag{12}$$

Since no general study can be done on (12), the next section presents results of $\varepsilon^2 - \varepsilon_0^2$ for different jitter statistics and for different spectral densities $s_Z(\omega)$.

3. SIMULATION RESULTS

Four kinds of spectra have been considered, as illustrated on Fig. 1. All spectral densities are function of a shape parameter a which can be related to an aliasing factor. The jitter random variables are assumed to be i.i.d. Different distributions have been studied including Gaussian, Laplacian, Bernoulli and uniform distributions. In most cases, $\varepsilon^2 - \varepsilon_0^2$ remains positive whatever the different investigated spectra and jitter statistics. Figure 2 illustrates the shape of $\varepsilon^2 - \varepsilon_0^2$ for a "Gaussian" power spectral density (PSD) – Fig. 1 (d) – and for a Gaussian jitter distribution with a zero mean and a standard deviation λ. The parameter a of the PSD is linked to the spectral bandwidth: the more it increases, the more the aliasing effect is present. However, this difference does not always increase as the jitter effect increases, as Fig. 2 seems to show. Figure 3 displays the

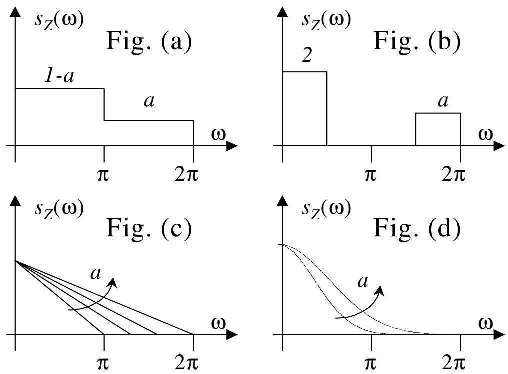

Fig. 1. Examples of spectral densities.

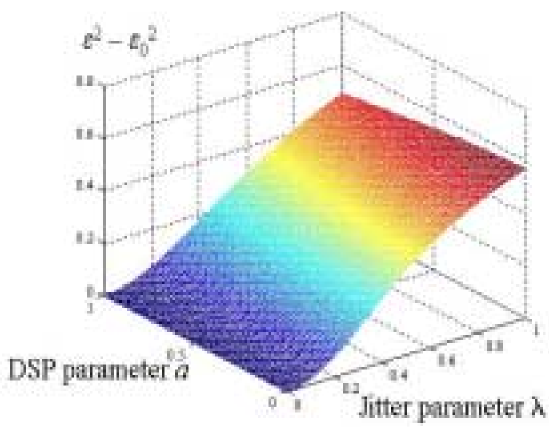

Fig. 2. $\varepsilon^2 - \varepsilon_0^2$ as a function of the spectral bandwidth a in case of a "Gaussian" PSD and of the Gaussian jitter standard deviation λ.

simulation result of $\varepsilon^2 - \varepsilon_0^2$ in case of a spectral density corresponding to Fig. 1 (a) with a Bernoulli jitter distribution with probability $\frac{1}{2}$. The parameter λ of the jitter distribution corresponds to the values taken by the binary jitter random variable: $P\left[A_n = \lambda\right] = P\left[A_n = -\lambda\right] = \frac{1}{2}$. Moreover, in some cases, $\varepsilon^2 - \varepsilon_0^2$ becomes negative. This has been observed mostly for the spectral density of Fig. 1 (b) with several kinds of jitter distributions and for a Laplacian jitter distribution, with several PSD shapes. Figure 4 presents the simulation result of $\varepsilon^2 - \varepsilon_0^2$ in case of a PSD such as in Fig. 1 (b) with a Laplacian jitter distribution with parameter $\lambda = 0.2$.

This result is surprising, even if (10) helps to understand why jitter can reduce aliasing. At first sight, it is hard to believe that introducing some information loss due to unobserved timing jitter helps to better recover the continuous-time process in the presence of aliasing. However, this paper shows that some examples can be given, such as the one of Fig. (4) where timing jitter can improve the continuous-time random process reconstruction when aliasing is present.

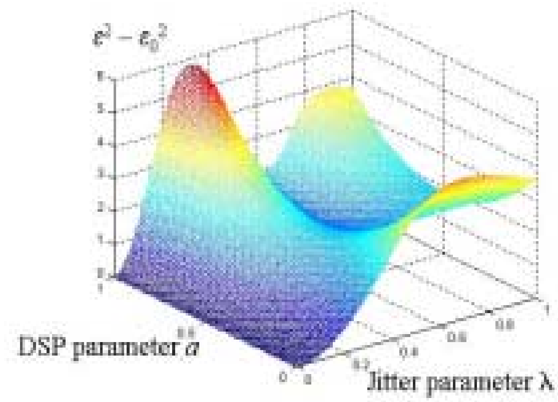

Fig. 3. $\varepsilon^2 - \varepsilon_0^2$ as a function of the spectral parameter a of the PSD of Fig.1(a) and of the Bernoulli distribution λ.

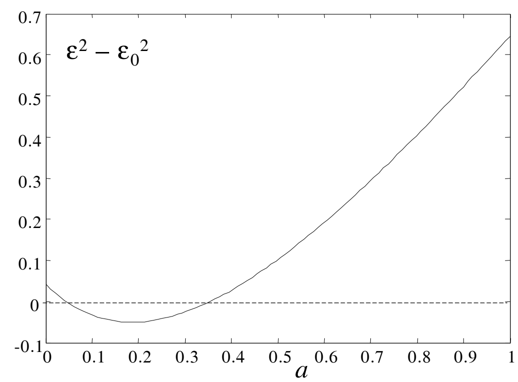

Fig. 4. $\varepsilon^2 - \varepsilon_0^2$ as a function of the spectral parameter a of the PSD of Fig.1(b) in case of a Laplacian jitter distribution.

4. CONCLUSION

The aim of this paper was to study the quality of the linear minimum mean square estimator (LMMSE) of a random process when the Nyquist condition is not fulfilled. The question under this study was to investigate whether some timing jitter could improve the quality of the LMMSE. At first sight, this result would be surprising since the timing jitter is unobserved and therefore yields a loss of information. However, the expression of the sampled process PSD in presence of aliasing and timing jitter is given in this paper. From this PSD expression, it is clear that timing jitter can in some cases reduce aliasing and in some other cases increase it. The expression of the mean reconstruction error has been derived with and without jitter. The general study of the difference between these two mean errors is impossible since it depends on the spectral density of the random process and on the jitter distribution. Therefore, simulations have been conducted in several cases. Different PSD shapes have been considered, as well as different jitter distributions. We have shown that, in most cases, jitter does not improve the quality of the LMMSE. However, some cases have been highlighted where the introduction of a timing jitter leads to a mean reconstruction error less than the one without jitter.

5. APPENDIX

The aim of this appendix is to derive the expression of the LMMSE of a random process $Z(t)$ when both aliasing and jitter effects are present. The observed samples are of the form given by (6) where the jitter is assumed to be modeled by a sequence of i.i.d. random variables A_n. Obviously, the jitter sequence is not observed and is only known by its characteristic functions (see (7)). The LMMSE $\hat{Z}(t)$ is defined as the orthogonal projection of $Z(t)$ on the Hilbert space $\mathbf{H}(\mathbf{U}_s)$ spanned by the observed samples U_n:

$$E\left[\left(Z(t)-\hat{Z}(t)\right)U_n^*\right]=0, \quad \forall n\in\mathbb{Z}. \tag{13}$$

Using conditional mathematical expectations yields

$$\begin{aligned} E\left[Z(t)\,U_n^*\right] &= E\left[Z(t)\,Z^*(n-A_n)\right] \\ &= E\left[E\left[Z(t)\,Z^*(n-a_n)\,/A_n=a_n\right]\right] \\ &= E\left[\int_{-\infty}^{+\infty} e^{i\omega t}e^{-i\omega(n-a(n))}s_Z(\omega)\,d\omega\right] \\ &= \int_{-\infty}^{+\infty} e^{i\omega(t-n)}\Psi(\omega)\,s_Z(\omega)\,d\omega. \end{aligned} \tag{14}$$

The same kind of computations leads to the expression of the autocorrelation function of the sampled process, using the independence of the random variables A_n:

$$\begin{aligned} E\left[U_mU_{m-n}^*\right] &= \int_{-\infty}^{+\infty} e^{i\omega n}\left|\Psi(\omega)\right|^2 s_Z(\omega)\,d\omega, \\ for \quad n &\neq 0 \end{aligned} \tag{15}$$

and

$$E\left[U_mU_m^*\right]=E\left[\left|Z(t)\right|^2\right]. \tag{16}$$

These two last equations allow to compute the power spectrum $s_{U_s}(\omega)$ defined on $(-\pi,\pi)$ of the sequence $\mathbf{U}_s$

$$s_{U_s}(\omega)=\sum_{k\in\mathbb{Z}} s_Z(\omega+2\pi k)\left|\Psi(\omega+2\pi k)\right|^2+c \tag{17}$$

where c is a constant given by

$$c=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(1-\left|\Psi(\omega)\right|^2\right)s_Z(\omega)\,d\omega. \tag{18}$$

Taking into consideration the general form of the LMMSE given in (2), and using (14), (13) can be written

$$\begin{aligned} &\textstyle\int_{-\infty}^{+\infty} e^{i\omega(t-n)}\Psi(\omega)\,s_Z(\omega)\,d\omega \\ &-\textstyle\int_{-\pi}^{+\pi}\mu_t(\omega)\,e^{-i\omega n}s_{U_s}(\omega)\,d\omega=0\;\forall n\in\mathbb{Z}. \end{aligned} \tag{19}$$

where $\mu_t(\omega)$ is defined in (3). If $\Psi(\omega)$ and $s_Z(\omega)$ are sufficiently regular, using the Fourier series unicity property, (19) leads to

$$\mu_t(\omega)=\frac{\sum_{k\in\mathbb{Z}} s_Z(\omega+2\pi k)\,\Psi(\omega+2\pi k)\,e^{i2\pi kt}}{s_{U_s}(\omega)}e^{i\omega t}. \tag{20}$$

The reconstruction error power σ_t^2 and its time average ε^2 defined in (11) are calculated using the Pythagorean theorem and the Plancherel equality

$$\sigma_t^2=E\left[\left|Z(t)\right|^2\right]-\int_{-\pi}^{+\pi}\left|\mu_t(\omega)\right|^2 s_{U_s}(\omega)\,d\omega \tag{21}$$

and

$$\begin{aligned} \varepsilon^2 &= E\left[\left|Z(t)\right|^2\right] \\ &\quad -\int_{-\pi}^{+\pi}\frac{\left(\sum_{k\in\mathbb{Z}}\left|\Psi(\omega+2\pi k)\right|^2 s_Z^2(\omega+2k\pi)\right)}{s_{U_s}(\omega)}d\omega. \end{aligned} \tag{22}$$

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7. REFERENCES

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