POLYPHASE ANALYSIS OF ALIASING EFFECTS IN ENLARGEMENTS

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ABSTRACT

Typical image processing applications use linear interpolation or piecewise cubic interpolation for resampling of images. These are popular since the interpolation kernels are small and the results are acceptable. However, since the frequency domain characteristics of the interpolation filters are not good, two effects usually appear and cause a noticeable degradation in quality of the image. The first is jagged edges and the second is low frequency modulation of high frequency components such as the sampling noise. Both effects result from aliasing. Enlargement of an image by a rational factor of (L/M) is represented by first interpolating the image on a grid L times finer than the original sampling grid, and then resampling it every M grid points. While the usual treatment of the aliasing is carried out by analyzing the interpolation filter in the frequency domain, this paper suggests analyzing the aliasing effects using a polyphase representation of the interpolation process. It turns out that the aliasing effects are caused by differences between the polyphase filters. We therefore define the average amplitude function and use it to measure the aliasing expected from the interpolation filter.

1. INTRODUCTION

The essence of this paper is about introducing a new efficient and simple approach for analyzing aliasing effects in enlargements, caused by the interpolation filter.

we consider the case where a separable interpolation kernel is used for resampling images, i.e., changing the number of pixels in the image. We concentrate on enlargement of images by a rational factor of L/M. Since the interpolation we discuss is separable, we conduct the analysis in one dimension and use the time and frequency domain. In section 2 we discuss the polyphase implementation of resampling. In section 3 we discuss the aliasing effects created by resampling. In section 4 we present the polyphase analysis for estimating the aliasing expected from the interpolation filter.

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2. RESAMPLING FORMULATION

In this section, we first introduce the resampling scheme in a similar manner as is discussed in [1]. This is followed by a polyphase representation as in [2], which is usually used in order to reduce computations. In our case this scheme is used for investigating the aliasing effects.

The resampling scheme is shown in Figure 1. The sequence $x[n]$ is input into an expander by L. Its output $x_I[n]$ is fed to the interpolation filter, having an impulse response $h[n]$. Note that in each group of L consecutive samples of $x_I[n]$, there are $L-1$ zeros. The interpolation filter "adds" the missing samples and so produces the interpolated $y_I[n]$. The interpolated signal $y_I[n]$ is then decimated by a factor of M to produce the output signal $y[n]$. A detailed analysis of this scheme appears in many textbooks, e.g., [1], [2].

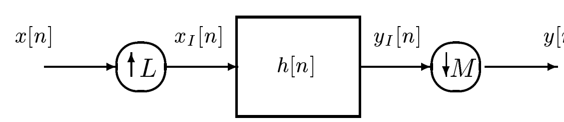

Fig. 1. Resampling by a rational factor of L/M

It is easy to show, that the expander and the interpolation filter can be replaced with the scheme depicted in Figure 2. This scheme, called a polyphase representation, is more efficient than the original scheme, since the number of multiplications is reduced by a factor of L. A polyphase representation is based on splitting $h[n]$ to L filters $h_l[n]$ where $l = 0, 1, \ldots, L-1$. The filter $h_l[n]$ is found by decimation, in a factor of L, of $h[h]$, shifted by l:

$$h_l[n] = h[nL + l] \tag{1}$$

Although there are more efficient schemes, [2], this scheme is perfectly suitable for our analysis.

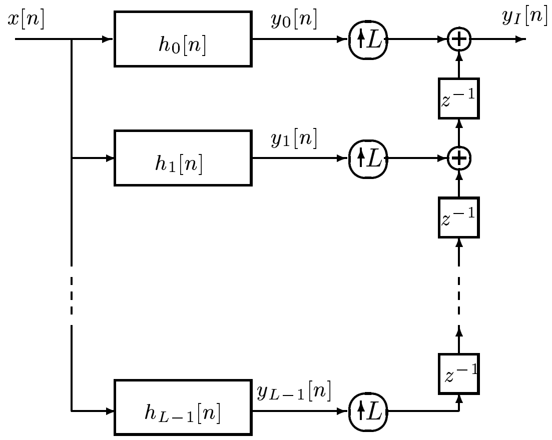

Fig. 2. The interpolation part using polyphase filters

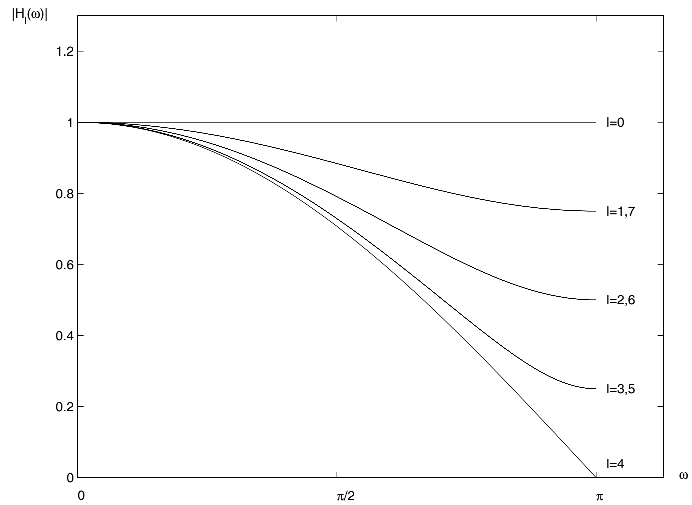

Fig. 3. $|H_l(\omega)|$-s of linear interpolation ($L = 8$)

3. ALIASING IN ENLARGEMENTS

In image processing, we prefer to have a small number of non zero coefficients in the sequence $h[n]$, i.e., a short sequence. This results with a short processing time. Common interpolation functions are: zero order hold, linear interpolation [3], and piecewise cubic interpolation, [4],[5]. These functions span along a small number of samples, resulting with short sequences $h_l[n]$. These interpolation filters were determined according to some "smoothness" requirements in the time domain. Unfortunately, $H(\omega)$, The Discrete-Time Fourier Transform (DTFT) of such interpolation filters is not bandlimited to $\omega \in [-\pi/L, \pi/L]$, and so we have aliasing effects after the decimation by M.

The conventional approach is therefore to improve $H(\omega)$ so the aliasing is reduced. However, it is difficult to evaluate the influence of changing $H(\omega)$ on the aliasing, looking at $H(\omega)$ directly. In this paper, we suggest a different approach to analyze the aliasing. We claim that the aliasing effects are easily and better understood when the polyphase representation described earlier is used. We consider the L outputs of the L $h_l[n]$ polyphase filters as L different sequences, denoted $y_l[n]$, and analyze them separately.

To demonstrate the reasoning behind our approach we chose to use the linear interpolation. The interpolation function $h[n]$ is given by:

$$h[n] = \begin{cases} 1 - |\frac{n}{L}| & 0 \leq |\frac{n}{L}| < 1 \\ \\ 0 & 1 \leq |\frac{n}{L}| \end{cases} \qquad (2)$$

The $h_l[n]$-s are calculated from $h[n]$ using equation(1).

In Figure 3, we show the absolute value of the DTFT of the filters $h_l[n]$, denoted $H_l(\omega)$, for $l = 0, \ldots, 7$. It is clear that these L filters have different frequency responses. This is so since the $h_l[h]$ are decimated versions of $h[n]$, shifted by l. Each of them is therefore affected differently by the decimation (from $h[n]$ to $h_l[n]$), and the aliasing in each of them, caused by that decimation, results with a considerably different shape of $|H_l(\omega)|$ for different l-s. Thus, the L signals $y_l[n]$ are influenced differently by the filters $H_l(\omega)$.

Actually, *the different shapes of the $H_l(\omega)$-s is the reason of having aliasing effects* .

The differences between the $|H_l(\omega)|$-s cause two main effects which are easily noticeable in images, and easily understood by the polyphase approach. One is jagged edges and the second is noise modulation. Let us explain the effects using Figures 2 and 3.

We first discuss the noise modulation. Say we have an image with a constant level. During the image acquisition, a wide band noise is added to that constant level by the A/D circuit or the sensor noise. When enlarging the image by a factor of say $16/15$, i.e., $L = 16$ and $M = 15$, we actually use all of the 16 filters $h_l[n]$ to calculate 16 consecutive output samples "from" an interval of 15 consecutive input samples and their neighbors. Thus, the output samples are taken from 16 images (sequences) $y_l[n]$ that are produced each by a different filter $H_l(\omega)$. Since those are different, we get a low frequency modulation of the noise. In Figure 4, we demonstrate this phenomenon on a small part of the Lena image to which we added white Gaussian noise prior to enlarging it by $16/15$ using linear interpolation. (The image was sharpened afterwards, in order to intensify the modulation effect, so it is easy to observe in print).

The other noticeable effect, is jagged edges. Say we enlarge an image by a factor of 2, i.e., $L = 2$ and $M = 1$. In this case the even samples of the output signal are duplicates of the input samples, while the odd output samples are "interpolated" by averaging the two adjacent input samples. Thus, the high frequency content of the odd pixels is

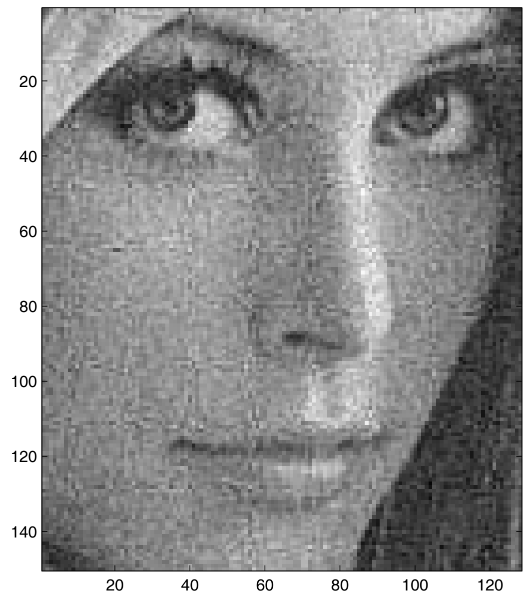

Fig. 4. Noise modulation produced by linear interpolation, ($L = 16$, $M = 15$)

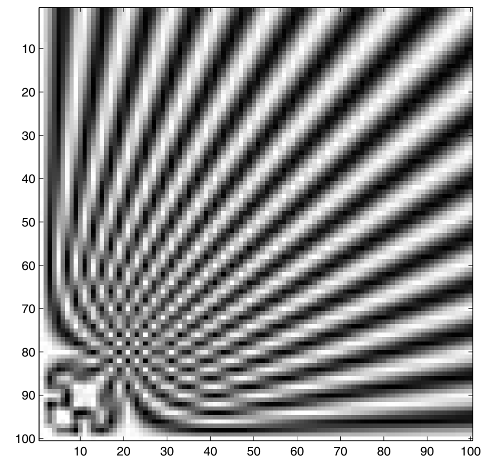

Fig. 5. Jagged edges in enlargement by 2 ($L = 2$, $M = 1$) using linear interpolation

blurred. In case we have an area in the image in which we have high frequencies, e.g., edges, this difference between the odd and even pixels is noticeable. The even pixels have more contrast than the odd blurred pixels. This is easily seen in edges that are almost horizontal or vertical, as in Figure 5 below. This figure is an enlargement of an image by a factor of 2, in the x and y directions, carried out by linear interpolation.

Again, the two effects described above are caused by the very same reason, *differences between the interpolation filters* $H_l(\omega)$.

4. POLYPHASE ANALYSIS OF ALIASING

We could define the desired interpolation filter $h[n]$ by specifying its frequency response $H(\omega)$. In case it is bandlimited to $[-\pi/L, \pi/L]$, the L polyphase filters, $H_l(\omega)$, have the same amplitude response, denoted $A(\omega)$, and their frequency responses are given by

$$H_l(\omega) = A(\omega)e^{j\omega l} \qquad (3)$$

Since all of the polyphase filters have the same amplitude and the appropriate phase, we expect no aliasing. It is easy to see, that in such a case, $H(\omega)$ is given by

$$H(\omega) = \begin{cases} L \cdot A(\omega L) & 0 \le |\omega| \le \frac{\pi}{L} \\ 0 & \text{otherwise} \end{cases} \qquad (4)$$

thus, it is really band limited, and so, no aliasing occurs.

We here suggest a measure, the Interpolation Error Index, denoted $F_2^{(L)}$, for the quality of an interpolation filter $h[n]$. The interpolation error index is defined as the average deviation of $H_l(\omega)$ from the desired filter of equation (3):

$$F_2^{(L)} = \frac{1}{2\pi L} \sum_{l=0}^{L-1} \int_{-\pi}^{\pi} |A(\omega)e^{j\omega l/L} - H_l(\omega)|^2 d\omega \qquad (5)$$

We denote the amplitude $A(\omega)$ that brings the error index $F_2^{(L)}$ to a minimum for a given set of filters $H_l(\omega)$, by $A_{av}^{(L)}(\omega)$. It is easy to see, by differentiating $F_2^{(L)}$ in respect to $A(\omega)$ that $A_{av}^{(L)}(\omega)$ is given by

$$A_{av}^{(L)}(\omega) = \frac{1}{L} \sum_{l=0}^{L-1} H_l(\omega)e^{-j\omega l/L} \qquad (6)$$

We refer to $A_{av}^{(L)}(\omega)$ as the average amplitude function of the polyphase filters. It is easy to see that when $h[n]$ is symmetric, i.e., $h[n] = h[-n]$, then $A_{av}^{(L)}(\omega)$ is real.

Using $A_{av}^{(L)}(\omega)$, we define the Aliasing Index $F_a^{(L)}$ as

$$F_a^{(L)} = \frac{1}{2\pi L} \sum_{l=0}^{L-1} \int_{-\pi}^{\pi} |A_{av}^{(L)}(\omega)e^{j\omega l/L} - H_l(\omega)|^2 d\omega \qquad (7)$$

We also define the Amplitude Index, $F_d^{(L)}(\omega)$, which indicates the deviation of the average amplitude function $A_{av}^{(L)}(\omega)$ from the desired amplitude function $A(\omega)$, as

$$F_d^{(L)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |A(\omega) - A_{av}^{(L)}(\omega)|^2 d\omega \qquad (8)$$

It is easy to see that these indices satisfy

$$F_2^{(L)} = F_a^{(L)} + F_d^{(L)} \qquad (9)$$

We now have separate measures for the aliasing and for the deviation of the average amplitude from the desired amplitude function.

So, now we can be more accurate and state that *the difference of the $H_l(\omega)$-s, (with a phase factor of $e^{-j\omega l/L}$), from $A_{av}^{(L)}(\omega)$, the average of the polyphase filters, is the reason of having aliasing effects.*

$F_a^{(L)}$ measures the difference from $A_{av}^{(L)}(\omega)$, but different interpolation filters have different $A_{av}^{(L)}(\omega)$-s. Thus, for a correct comparison of interpolation filters, we should "equalize" or "normalize" the interpolation filter so that the normalized polyphase filters have $A_{av}^{(L)}(\omega) = 1$. Therefore, we define the Normalized Aliasing Index, $F_A^{(L)}$ as

$$F_A^{(L)} = \frac{1}{2\pi L} \sum_{l=0}^{L-1} \int_{-\pi}^{\pi} \left| \frac{A_{av}^{(L)}(\omega) e^{j\omega l/L} - H_l(\omega)}{A_{av}^{(L)}(\omega)} \right|^2 d\omega \tag{10}$$

This is equivalent to a series LPF equalizer, with a frequency response of $1/A_{av}^{(L)}(\omega L)$ for $|\omega| < \pi/L$, which equalizes the average amplitude to 1.

Let us apply the Normalized Aliasing and the Amplitude Indices to a few common interpolation functions. We assume that the desired amplitude function $A(\omega)$ equals 1, as in an ideal LPF, and do the comparison for $L = 8$.

We start with the simple zero order hold interpolation, also called nearest neighbor interpolation, and given by

$$h[n] = \begin{cases} 1 & -0.5 \leq \frac{n}{L} < 0.5 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

For $A(\omega) = 1$, we find $F_A^{(8)} = 0.2873$, $F_d^{(8)} = 0.0373$ ($F_2^{(8)}$ is 0.2599).

For linear interpolation, also called a first order hold interpolation, described by equation (2), we have $F_A^{(8)} = 0.0661$, $F_d^{(8)} = 0.0833$ ($F_2^{(8)}$ is 0.1182).

For the 4 points cubic piecewise interpolation of [4],[5], also called third-order cubic interpolation, [6], we find $F_A^{(8)} = 0.0611$, $F_d^{(8)} = 0.0401$ ($F_2^{(8)}$ is 0.0779).

Finally, for the 6 points cubic piecewise interpolation of [5], also called fourth-order cubic interpolation, [6], we find $F_A^{(8)} = 0.0495$, $F_d^{(8)} = 0.0329$ ($F_2^{(8)}$ is 0.0635).

From looking at the DTFT-s of those interpolation filters and from applying them to actual images, it seems that the Normalized Aliasing Index is a good measure for estimating aliasing effects, while the Amplitude Index is a good measure for high frequency degradation.

When two interpolation filters have equal Normalized Aliasing Indices, this suggests that they produce a similar amount of aliasing effects. Therefore, we may use the Normalized Aliasing Index for comparing aliasing of interpolation filters. Note that when comparing interpolation filters, we should give the appropriate attention to both the Normalized Aliasing Index and the Amplitude Index.

5. CONCLUSION

Aliasing in enlargement of images has two main effects, periodic modulation of noise and jagged edges. The classical solution to this problem is to improve the interpolation filter. However, we lack a systematic analysis of the quality of that LPF in terms of aliasing effects. In this paper we introduced a new approach for aliasing analysis in enlargements, based on looking at the polyphase representation of the interpolation process. We explained that aliasing effects result from differences between the frequency responses of the polyphase filters, and defined the Normalized Aliasing Index, which measures the aliasing expected from an interpolation filter based on deviation from the average amplitude of the polyphase filters. We also defined the Amplitude Index, measuring the deviation of the interpolation filter from the desired amplitude response. These indices were found to agree with the aliasing and the high frequency degradation we have when applying common interpolation filters on images.

Since the aliasing effects result from the differences between the frequency responses of the polyphase filters, equalizing those filters reduces the aliasing. The new measures introduced in this paper can be used for evaluation of the quality resulting after such equalization.

6. REFERENCES

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