TRANSMIT SIGNAL DESIGN FOR OPTIMAL DECONVOLUTION[1]

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ABSTRACT

This paper solves the problem of designing successive transmit pulses to minimize the mean squared error of deconvolution in colored noise. The power spectral densities of the signal and noise are known *a priori*. In the special case of white noise, the analytical solution can be interpreted as the "water-filling" principle for the distribution of transmitted spectral energy. Numerical simulation of a simple example with exponentially correlated target signal and white noise illustrates the benefits of using adaptively designed transmit signals.

1. INTRODUCTION

Deconvolution using Wiener filters has broad applications in communications (channel estimation and equalization), radar and sonar (target imaging), image processing (deblurring and restoration), etc. The Wiener filter is optimal in the minimum mean square error (MMSE) sense. This optimal deconvolution solution can be found in many statistical signal processing textbooks (e. g. [1]), however, nearly all assume a fixed and known convolution kernel. In applications such as radar and sonar, the convolution kernel is the transmitted waveform, while the signal to be deconvolved is the target range profile. Therefore adjusting the convolution kernel (subject to certain constraints) in order to achieve the best deconvolution is feasible, at least conceptually.

There has been extensive work on the design of transmit signal which optimizes detection performance of point target in signal-dependent interference and ambient noise [2,3,4,5]. A recent work by Pillai *et. al.* [6] considered non-point targets with the objective of maximizing the signal to interference ratio (SINR). None have formulated the problem as minimizing the estimation error of target profile. Our work here uses the Wiener filter to reconstruct the target profile in the MMSE sense, and the transmit signal is optimized (subject to pulse energy constraint) to further minimize the mean square error achieved by the Wiener filter. Our model assumes *a priori* knowledge of the target and noise in the form of their power spectral densities (PSD), or equivalently their autocorrelation functions. We further extend our work to multiple pulse transmission and deconvolution in that, after each pulse, the *a posteriori* PSD is calculated and used as the *a priori* PSD of the next pulse. This is in contrast to traditional approach of transmitting multiple identical pulses, and will reduce the deconvolution error more rapidly in low SNR situation.

In the remainder of this paper, we shall present analytical derivations of optimal transmit signal design for single and multiple transmissions, accompanied by numerical simulations that demonstrate the advantages of the optimal signals over traditional broadband signals. In the final section, we discuss limitations and possible extensions of the current work, and a potential relationship of this work to a result in information theory.

2. ANALYTICAL SOLUTION FOR SINGLE PULSE

Our model is the standard linear time invariant system with additive Gaussian noise:

$$y[n] = (h * x)[n] + v[n] \tag{1}$$

where $y[n]$ is the received signal, $h[n]$ is the transmit signal to be designed, $x[n]$ is the target signal with PSD $S_{XX}(\omega)$, and $v[n]$ is noise with PSD $S_{VV}(\omega)$. All signals are zero mean Gaussian. Our task is to design $h[n]$ to achieve the minimum mean squared error between $x[n]$ and the $\hat{x}[n]$, which is estimate of $x[n]$. To make the problem well-posed, we impose the unit energy constraint on $h[n]$. Therefore the design problem is mathematically stated as:

$$\min_{h[n]} E\left\{\left|\hat{x}[n] - x[n]\right|^2\right\} \tag{2}$$

subject to

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right|^2 = 1 \tag{3}$$

In the following, we solve this optimization problem in the frequency domain. We know that for a fixed $h[n]$ the MMSE estimator is the (non-causal) Wiener filter:

$$F(\omega) = \frac{H^*(\omega) S_{XX}(\omega)}{\left|H(\omega)\right|^2 S_{XX}(\omega) + S_{VV}(\omega)} \tag{4}$$

and the mean squared error (MSE) attained by the Wiener filter is [7]

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$$\varepsilon = E\left\{\left|\hat{x}[n]-x[n]\right|^2\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{S_{VV}(\omega)S_{XX}(\omega)}{\left|H(\omega)\right|^2 S_{XX}(\omega)+S_{VV}(\omega)}d\omega \quad (5)$$

Let $G(\omega)=\left|H(\omega)\right|^2$. The constrained optimization problem is then transformed to

$$\min_{G(\omega)}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{S_{VV}(\omega)S_{XX}(\omega)}{G(\omega)S_{XX}(\omega)+S_{VV}(\omega)}d\omega \quad (6)$$

subject to
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}G(\omega)d\omega = 1 \quad (7)$$

and
$$G(\omega)\geq 0 \quad (8)$$

The Kuhn-Tucker (first order necessary) conditions [8] for this minimization problem are that there exist constant λ and function $\mu(\omega)\leq 0$ such that:

$$\frac{-S_{XX}^2(\omega)S_{VV}(\omega)}{\left[G(\omega)S_{XX}(\omega)+S_{VV}(\omega)\right]^2}+\lambda+\mu(\omega)=0 \quad (9)$$

and
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mu(\omega)G(\omega)d\omega = 0 \quad (10)$$

Two remarks are due here. First, it is easy to verify that constraints (7) and (8) are convex and the objective function (6) over the constraint set is also convex. Therefore the Kuhn-Tucker conditions are also sufficient. Second, since $\mu(\omega)\leq 0$ and $G(\omega)\geq 0$, equation (10) implies $\mu(\omega)G(\omega)=0$, thus for every ω either $\mu(\omega)=0$ or $G(\omega)=0$. From (9) we obtain

$$G(\omega)=\sqrt{\frac{S_{VV}(\omega)}{\lambda+\mu(\omega)}}-\frac{S_{VV}(\omega)}{S_{XX}(\omega)} \quad (11)$$

where λ and $\mu(\omega)$ are to be determined from the constraints. Substituting (11) into (7), and noting that $\mu(\omega)=0$ whenever $G(\omega)\neq 0$, we have

$$\begin{aligned} 1 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sqrt{\frac{S_{VV}(\omega)}{\lambda}}-\frac{S_{VV}(\omega)}{S_{XX}(\omega)}\right]u\left(\sqrt{\frac{S_{VV}(\omega)}{\lambda}}-\frac{S_{VV}(\omega)}{S_{XX}(\omega)}\right)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sqrt{\frac{S_{VV}(\omega)}{\lambda}}-\frac{S_{VV}(\omega)}{S_{XX}(\omega)}\right]u\left(\frac{S_{XX}^2(\omega)}{S_{VV}(\omega)}-\lambda\right)d\omega \end{aligned} \quad (12)$$

where $u(\cdot)$ is the unit step function. Eq. (12) should be used to numerically determine λ. After λ is determined, the solution for the optimal $G(\omega)$ is:

$$G(\omega)=\left[\sqrt{\frac{S_{VV}(\omega)}{\lambda}}-\frac{S_{VV}(\omega)}{S_{XX}(\omega)}\right]u\left(\frac{S_{XX}^2(\omega)}{S_{VV}(\omega)}-\lambda\right) \quad (13)$$

The mean square error achieved by this solution can then be obtained by substituting (13) into (5):

$$\varepsilon=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[S_{XX}(\omega)+\left(\sqrt{\lambda S_{VV}(\omega)}-S_{XX}(\omega)\right)u\left(\frac{S_{XX}^2(\omega)}{S_{VV}(\omega)}-\lambda\right)\right]d\omega \quad (14)$$

It can be easily shown that

$$\varepsilon<\frac{1}{2\pi}\int_{-\pi}^{\pi}S_{XX}(\omega)d\omega = E\left\{\left|x[n]\right|^2\right\} \quad (15)$$

which is expected since deconvolution should reduce the uncertainty of the target signal. Note that the optimal transmit signal is given in terms of its spectral power. Any time domain signal $h[n]$ that satisfies $\left|H(\omega)\right|^2=G(\omega)$ is an optimal solution. This is reasonable since the target $x[n]$ is characterized only by its PSD $S_{XX}(\omega)$, therefore only the spectral power of the excitation pulse matters.

Now let us consider the special case of white noise, i.e. $S_{VV}(\omega)=\sigma^2$. The solution in (13) can be rewritten as

$$G(\omega)=\left[C-\frac{\sigma^2}{S_{XX}(\omega)}\right]u\left(C-\frac{\sigma^2}{S_{XX}(\omega)}\right) \quad (16)$$

where $C=\sigma^2/\sqrt{\lambda}$, and C should be determined by the energy constraint in (7). The solution in (16) can be interpreted using the "water-filling" analogy: Imagine a "spectral reservoir" with a bottom profile of $\sigma^2/S_{XX}(\omega)$. If we fill this "reservoir" with the available energy "fluid", then the "fluid" will seek a constant level C and provide the optimal spectral energy distribution (figure 1). Frequencies with high $\sigma^2/S_{XX}(\omega)$ values (low SNR) receive less and possibly no energy. On the other hand, if $S_{XX}(\omega)$ is white, then Eq. (16) implies $G(\omega)=$ constant, indicating that a broadband (spectrally flat) pulse (e.g. a chirp) is the optimal transmit signal in this case.

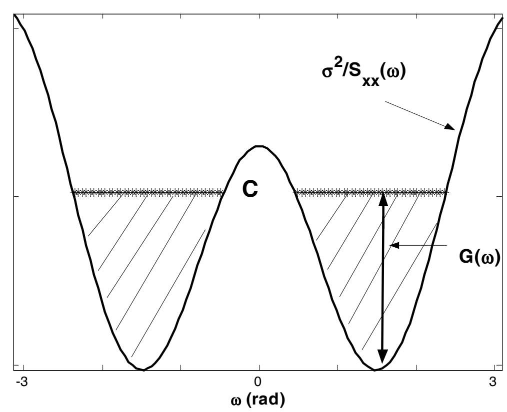

Figure 1: Water-filling distribution of spectral energy

3. DESIGN OF MULTIPLE-PULSE SEQUENCE

Suppose we send a sequence of N pulses $h^{(k)}[n]$, $k=1,2,\ldots,N$, to estimate the target signal. The traditional approach is to send N identical pulses, resulting in a final MSE that is 1/*N* of that using only one pulse. However, one could view this as a successive estimation problem, where after each pulse the *a posteriori* statistics of the target signal is calculated and then used as the *a priori* statistics for the design of the next pulse. In this section we propose such an algorithm to design the adaptive pulse sequence. Specifically, let $h^{(k)}[n]$ and $y^{(k)}[n]$ be the k'th transmit and received signal respectively, i.e., $y^{(k)}[n]=(h^{(k)}*x)[n]+v^{(k)}[n]$, and let $\mathbf{y}^{(k)}$ be the vector notation of $y^{(k)}[n]$. Define the *residual* target signal $x^{(k)}[n]$ recursively as $x^{(0)}[n]=x[n]$, $x^{(k+1)}[n]=x^{(k)}[n]-\hat{x}^{(k)}[n]$, where $\hat{x}^{(k)}[n]$ is the MMSE (Wiener filter) estimate of $x^{(k)}[n]$ given $k+1$ received pulses $\{\mathbf{y}^{(k+1)},\mathbf{y}^{(k)},\ldots,\mathbf{y}^{(1)}\}$. It is easy to see that for any k we have $x[n]=x^{(k)}[n]+\sum_{i=0}^{k-1}\hat{x}^{(i)}[n]$, and the MMSE estimate of $x[n]$ given k received pulses is $\hat{x}[n]=\sum_{i=0}^{k-1}\hat{x}^{(i)}[n]$. Let us also define

$$z^{(k)}[n]=y^{(k)}[n]-\sum_{i=0}^{k-1}\left(h^{(k)}*\hat{x}^{(i)}\right)[n] \tag{17}$$

then $z^{(k)}[n]=(h^{(k)}*x^{(k)})[n]+v^{(k)}[n]$. The estimate of $x^{(k)}[n]$ given $y^{(k)}[n]$ is the same as given $z^{(k)}[n]$. Now we can design $h^{(k)}[n]$ to optimally estimate $x^{(k)}[n]$ using Wiener filter. Let $S_{XX}^{(k)}(\omega)$ be the *a priori* PSD of $x^{(k)}[n]$ before receiving the $k+1$'th pulse, or equivalently the *a posteriori* PSD given k received pulses. In particular, $S_{XX}^{(0)}(\omega)=S_{XX}(\omega)$. Then the k+1'th pulse should be designed by, according to (13),

$$G^{(k+1)}(\omega)=\left[\sqrt{\frac{S_{VV}(\omega)}{\lambda^{(k+1)}}}-\frac{S_{VV}(\omega)}{S_{XX}^{(k)}(\omega)}\right]u\left(\frac{\left[S_{XX}^{(k)}(\omega)\right]^2}{S_{VV}(\omega)}-\lambda^{(k+1)}\right) \tag{18}$$

After receiving the k+1'th pulse, the MMSE estimate of $x^{(k)}[n]$ is obtained by Wiener filter. The residual error signal $x^{(k+1)}[n]$ has PSD [7]:

$$S_{XX}^{(k+1)}(\omega)=\frac{S_{VV}(\omega)S_{XX}^{(k)}(\omega)}{G^{(k+1)}(\omega)S_{XX}^{(k)}(\omega)+S_{VV}(\omega)} \tag{19}$$

where $G^{(k)}(\omega)\equiv\left|H^{(k)}(\omega)\right|^2$. Therefore, Eqs. (18) and (19) can be used to recursively compute $S_{XX}^{(k)}(\omega)$ and $G^{(k)}(\omega)$.

The complete adaptive transmit signal design and deconvolution algorithm can then be summarized as:

1. Initialize. $k=1$, $x^{(0)}[n]=x[n]$, $S_{XX}^{(0)}(\omega)=S_{XX}(\omega)$.
2. Design $G^{(k)}(\omega)$ by (18), where $\lambda^{(k)}$ is determined by solving $\frac{1}{2\pi}\int_{-\pi}^{\pi}G^{(k)}(\omega)d\omega=1$.
3. Construct any $h^{(k)}[n]$ such that $G^{(k)}(\omega)=\left|H^{(k)}(\omega)\right|^2$. Transmit $h^{(k)}[n]$ and receive $y^{(k)}[n]$. Compute $z^{(k)}[n]$ by (17).
4. Compute $\hat{x}^{(k-1)}[n]$ by Wiener filtering $z^{(k)}[n]$ with $f^{(k)}[n]$, where $F^{(k)}(\omega)=\frac{H^{(k)^*}(\omega)S_{XX}^{(k-1)}(\omega)}{G^{(k)}(\omega)S_{XX}^{(k-1)}(\omega)+S_{VV}(\omega)}$ is the frequency response of the Wiener filter.
5. Compute $\hat{x}[n]=\sum_{i=0}^{k-1}\hat{x}^{(i)}[n]$. This is the MMSE estimate given k received pulses.
6. Compute residual signal PSD $S_{XX}^{(k)}(\omega)$ by (19).
7. Compute the MSE given k received pulses. $\varepsilon^{(k)}=\frac{1}{2\pi}\int_{-\pi}^{\pi}S_{XX}^{(k)}(\omega)d\omega$.
8. $k=k+1$. Go to 2.

If the noise is white with $S_{VV}(\omega)=\sigma^2$, then Eq. (19) leads to

$$\frac{\sigma^2}{S_{XX}^{(k+1)}(\omega)}=G^{(k+1)}(\omega)+\frac{\sigma^2}{S_{XX}^{(k)}(\omega)} \tag{20}$$

This equation has the simple solution

$$\frac{\sigma^2}{S_{XX}^{(k)}(\omega)}=\frac{\sigma^2}{S_{xx}(\omega)}+\sum_{i=0}^{k-1}G^{(i)}(\omega),\quad k\geq 1 \tag{21}$$

It immediately follows that

$$S_{XX}^{(k)}(\omega)=\frac{\sigma^2 S_{XX}(\omega)}{S_{XX}(\omega)\sum_{i=0}^{k-1}G^{(i)}(\omega)+\sigma^2} \tag{22}$$

This is the same result as deconvolution using a single pulse whose spectral power is $G(\omega)=\sum_{i=0}^{k-1}G^{(i)}(\omega)$. Thus we conclude that $\sum_{i=0}^{k-1}G^{(i)}(\omega)$ "water–fills" the "reservoir" of $\sigma^2/S_{XX}(\omega)$ with total energy k.

4. NUMERICAL EXAMPLE

In this section we give a numerical example of deconvolving an exponentially correlated target signal in white noise. The signal has autocorrelation function $r_{xx}[n]=\alpha^{|n|}$, $|\alpha|<1$, corresponding to a PSD

$$S_{XX}(\omega)=\frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos\omega} \tag{23}$$

The noise is white with PSD $S_{VV}(\omega)=\sigma^2$.

To illustrate the advantage of using optimally designed transmit signal sequence over the classical approach of using multiple identical broadband signals, we plot the MSE of the estimate versus pulse number for both approaches. Figure 2 shows the results for different values of α and σ^2.

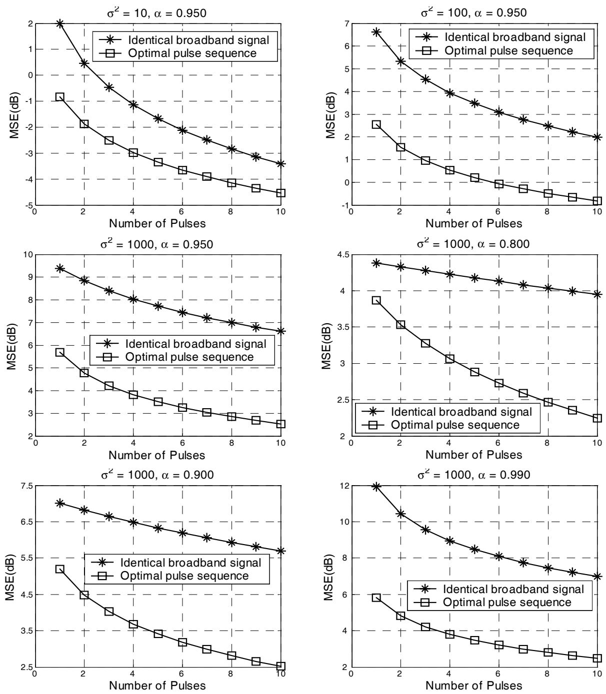

Figure 2: MSE reduction for different values of α and σ^2.

From these results we can make the following observations. First the MSE of the optimally designed transmit signal is always smaller than that of a broadband signal, as expected. Next, the performance gain is higher for bigger σ^2 (lower SNR). This is reasonable since at very high SNR, the optimal transmit signal approaches a broadband signal. Thirdly, the performance gain is higher for $|\alpha|$ closer to one. This is because small $|\alpha|$ implies less correlated target signal and therefore whiter PSD. If the target signal PSD is completely white, then the optimal transmit signal is again a broadband white signal, and there is no advantage of the optimally designed signal.

5. CONCLUSIONS

We have derived an analytical solution for designing transmit signal to minimize the deconvolution mean square error using Wiener filter. In the case of white noise, the solution is the "water-filling" distribution of the spectral energy of the transmit signal. An adaptive pulse selection scheme is proposed to design a pulse sequence that successively minimizes estimation error. Numerical simulation shows that the optimized transmit signal improves estimation performance significantly, particularly in low SNR environments. The current formulation is limited to non-causal Wiener filter. This is fine since in radar and sonar range imaging the deconvolution does not begin until the complete echo of a transmitted pulse is received. Formulation for causal Wiener filter would be interesting but seems significantly harder. A possible extension of this work is to include additional time domain constraints of the transmit signal, such as the length of $h[n]$. Another potential extension is application to array design for optimal coherent imaging of random media.

It is well known in communications theory that for vector Gaussian channel with white noise, the optimal energy distribution maximizing the mutual information follows the water-filling principle [9]. This implies that if we were to design a transmit signal to achieve the channel capacity, given the channel (target signal) PSD, we would have obtained the same "water-filling" solution as in (16). We believe that this is not a coincidence but rather a more fundamental relationship. However, this relationship between MMSE deconvolution and channel capacity has not been formally established in the literature, and is worthy of further theoretical development.

6. REFERENCES

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