JOINT AR PARAMETER AND ORDER ESTIMATION IN A GENERAL NOISE ENVIRONMENT

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ABSTRACT

A new joint autoregressive (AR) parameter and order estimation approach is presented which works well regardless of noise type. Unlike many other existing AR estimation methods which require prior knowledge of model order and noise statistics, the proposed method provides embedded order estimation in the pole-zero domain without requiring any noise statistics. The joint method virtually achieves the Cramor-Rao bound (CRB) in a general noise environment.

1. INTRODUCTION

The fundamental effect of white noise on AR signals is well understood in some pioneering papers [1][2], where the serious accuracy degradation due to the white noise is reported. Recent work on the noisy AR estimation problem can be found in the literature, e.g., [3]-[8]. In [3][4][5][8], an augmented AR model (i.e., AR(p+1)) plays an important role in estimating noise variance as well as AR parameters. In [6], the noise variance and AR parameters are computed by solving 2 sets of linear and nonlinear equations. The colored noise case is considered in [8] with 20dB SNR.

Although many papers have dealt with the noisy AR estimation problem, in most cases the noise is confined to be stationary and white and prior knowledge of the AR order is required. The computation of noise statistics is another burden for a reliable estimation result. To say nothing of nonstationary case, even with stationary colored noise, estimation becomes difficult and inaccurate. In this context, two challenging issues follow. The first is how to deal with the nonstationary noise. The second is how to do AR noisy estimation without separate AR order estimation. Some recent approaches for AR order estimation may be found in [9][10]. Most of the well-known criteria (e.g., AIC, MDL CAT, etc.) used in estimating AR order, however, may be used as only indicators of model order [11].

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In this paper, the two issues are effectively tackled in the pole-zero domain. The joint method does not require noise statistics or any classic model order criteria. We develop the key idea in the white noise case and then extend it to nonstationary colored noise. To our knowledge, the joint method is the first approach which is working reliably for AR order and parameter estimation regardless of SNR and noise statistics.

2. PROBLEM STATEMENT

A noisy AR(p) signal may be modeled as the series connection of an all-pole filter and noise addition as in Fig. 1, where $A_p(z) = 1+a_p(1)z^{-1} + \cdots + a_p(p)z^{-p}$ (say, $\mathbf{a}_p = [a_p(1), a_p(2), \cdots, a_p(p)]$) and the additive noise $v(n)$ constitute the model.

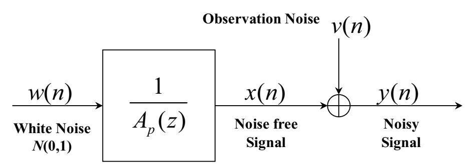


Fig. 1. AR(p) signal model with observation noise.

With an uncorrelatedness assumption on the signal and noise, we have a relationship of second order statistics,

$$\mathbf{R}_x = \mathbf{R}_y - \mathbf{R}_v \quad (1)$$

where $\mathbf{R}$ is the autocorrelation matrix. We may simply compute the AR parameters by $\mathbf{a}_p = -\mathbf{R}_x^{-1}\mathbf{r}_x$ where $\mathbf{r}_x = [r_x(1), r_x(2), \cdots, r_x(p)]^T$ and $r_x(k) = E[x(n)x(n-k)]$. However, finding a reliable estimate of $\mathbf{R}_v$ from $y(n)$ is not a trivial matter. Making it worse, the model order p is not known in general. $\mathbf{R}_v$ may be estimated indirectly (e.g., [8]) with limited accuracy. The fundamental limitation of depending on (1) happens especially when the noise is nonstationary. The main goal is to overcome such limitations.

3. JOINT ESTIMATION APPROACH

3.1. Mathematical Motivation

For the convenience of illustration, let $v(n)$ be stationary white noise with power σ_v^2. Then the power spectrum of the noisy AR signal $y(n)$ in Fig. 1 is readily expressed as

$$P_y(z) = \frac{1}{A_p(z)A_p(z^{-1})} + \sigma_v^2 = \frac{1+\sigma_v^2 A_p(z)A_p(z^{-1})}{A_p(z)A_p(z^{-1})}. \quad (2)$$

Letting $B_p(z)B_p(z^{-1}) = 1+\sigma_v^2 A_p(z)A_p(z^{-1})$, where $B_p(z) = b_p(0)+b_p(1)z^{-1} + \cdots + b_p(p)z^{-p}$, we see that $y(n)$ should be modeled as an ARMA(p,p) signal, i.e., $Y(z)=B_p(z)/A_p(z)$. Direct estimation of the ARMA(p,p) parameters is not appropriate because of poor accuracy and unknown order. We approximate the ARMA(p,p) signal to a higher order AR(s) signal. The overall noisy AR transfer function is then

$$\frac{B_p(z)}{A_p(z)} = \frac{1}{A_p(z)/B_p(z)} \approx \frac{1}{D_s(z)} \quad (3)$$

where $D_s(z)$ is an approximate polynomial with order s, i.e., $D_s(z)=d_s(0)+d_s(1)z^{-1}+ \cdots +d_s(s-1)z^{-(s-1)}+d_s(s)z^{-s}$. We may solve this approximate problem in an LS sense to show that the roots of $D_s(z)$ should include all roots of $A_p(z)$ and the remaining $s-p$ roots, which we call noise poles, are exclusively due to $B_p(z)$ in the middle of approximation, (3).

Let us consider nonstationary white noise. If σ_v^2 in (2) changes with time, this means that only $B_p(z)$ in (3) would change with time. In other words, the time varying σ_v^2 will affect only the noise poles of the AR(s) signal. In the colored case, the noise still has no control over the true poles whether it is stationary or not. But the colored noise must be finitely autocorrelated, i.e., an MA noise signal or having $r_v(k) = 0$ for any $|k|$ greater than a positive integer. Note that the noiseless AR(p) signal $x(n)$ then becomes an ARMA(p,q) model ($q > p$). The MA assumption on colored $v(n)$ actually excludes AR noise signals. The reason is that stationary AR noise adds extra poles and they will look like true poles in the pole-zero domain. Fortunately, the MA noise model encompasses a wide range of colored noise environments [8]. If the colored noise is not stationary, with any combinations of changing power, changing order, and changing coefficients, it will affect the MA part of $y(n)$ in the same manner as the white noise case.

3.2. The Joint Estimation Approach

According to the previous discussion, virtually no matter what the noise statistics are, the AR(p) parameters $\mathbf{a}_p = [a_p(1),a_p(2), \cdots, a_p(p)]^T$ are purely melted down to the AR(s) parameters $\mathbf{d}_s=[d_s(0),d_s(1),d_s(2), \cdots, d_s(s)]^T$ in some quite nonlinear manner. The key point is that the information would be revealed as p poles among s poles of the AR(s) signal. We select the true p poles via multiple test-runs and artificial noise perturbation as follows.

Since the true signal is usually assumed to have stronger self correlation than the noise signal, upon doing M test-runs in the pole-zero domain we can expect that there are exactly p strongly flocking groups and each group consists of M poles. Thus, we may simply count the number of groups to obtain the model order p and average each group to compute the AR(p) parameter estimate. Ironically and obviously, nonstationarity of the noise is more helpful in this joint estimation because it tends to scatter the noise poles more, so we want to make the MA noise part of $y(n)$ to be more nonstationary by the artificial noise perturbation before we do the actual estimation as shown in Fig. 2.

The artificial noise perturbation consists of two parts: the nonstationary white noise ($z(n)$ in Fig.2) generation and the average noise compensation. The artificial noise can be generated in many ways. In this paper, we set two noise variances defined by some ratios of the variance of $y(n)$ and then generate the first half of $z(n)$ to have one of the pre-set noise variances and the second half to have the other. Thus $y_1(n)$ in Fig. 2 contains a nonstationary noise part as well as the stationary AR(p) signal part. $z(n)$ is compensated by subtracting the average noise variance from the diagonal terms of the autocorrelation matrix of $y_1(n)$. Note that the purpose and meaning of the artificial noise compensation are fundamentally different from that in [1][2] where the purpose is to remove stationary white observation noise as exactly as possible to make a less noisy situation. In Fig. 2, however, we reduce the overall noise power to prevent unreasonably high noise power.

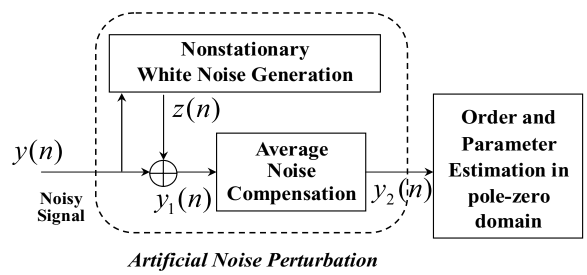


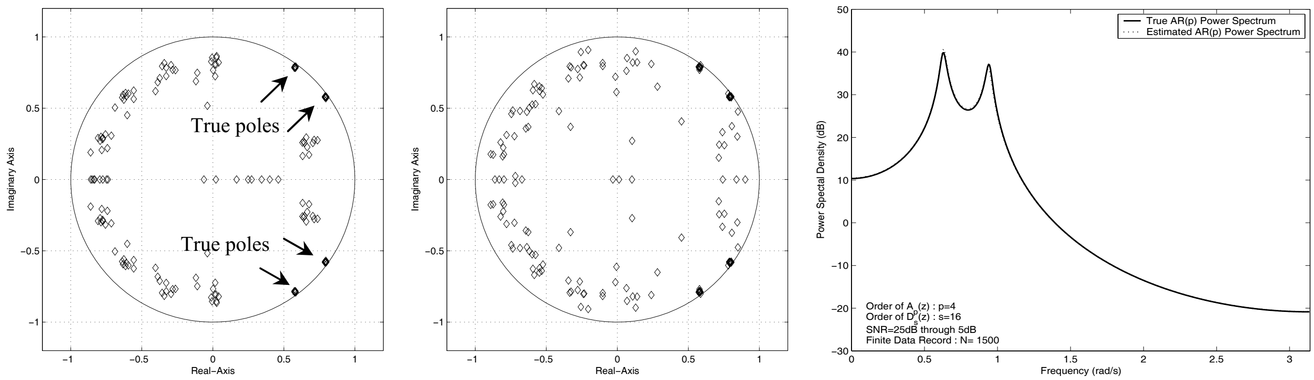
Fig. 2. Structure of the joint estimator.

3.3. On the Accuracy of AR Parameter Estimation

Since the approximation in (3) can be replaced with equality in case of infinite order s, the joint estimation based on the higher order AR(s) pole location is asymptotically unbiased. As shown in simulation results, however, s does not need to be very large. Another important performance criterion is the consistency, which usually requires small estimation variance as well. In general, it turns out that the CRBs of AR parameter estimates depend on the locations of the true poles [12]. In simulation, the joint method virtually achieves the theoretical lower bounds regardless of noise type.

4. SIMULATION RESULTS

We define an AR(4) process that has two pairs of poles at $z=0.98e^{\pm j(0.2\pi)}$ and $0.98e^{\pm j(0.3\pi)}$. The unknown observation noise is either stationary white, nonstationary white, stationary colored, or nonstationary colored. A colored noise signal is generated by passing zero mean white noise through an FIR filter defined as $H(z) = 1-1.0z^{-1}+0.2z^{-2}$ [8].



(a) (b) (c)

Fig. 3. Simulation results for nonstationary colored noise corrupted signal with s=16, N=1500, SNR=25 to 5dB, and M=10; Overlay pole-zero plot (a) without noise perturbation and (b) with noise perturbation, and (c) Reconstructed AR Spectrum.

We mention that the FIR filter is not restricted unless any zeros happen to cancel out the true poles. To impose nonstationarity, we change the noise variance corresponding to 25dB through 5dB SNR (abruptly or linearly). For comparison with other methods, we use the relative error (RE) and the averaged coefficient of variation (ACV) defined in [3] as

$$\mathrm{RE} = \frac{\left\| \text{sample mean vector of estimates} - \mathbf{a}_p \right\|}{\left\| \mathbf{a}_p \right\|},$$

$$\mathrm{ACV} = \frac{1}{p}\sum_{i=1}^{p} \frac{\sigma_t(\hat{a}_p(i))}{\left| \text{sample mean of estimates of } a_p(i) \right|},$$

where $\sigma_t(\hat{a}_p(i))$ is standard deviation (STD) of estimates $\hat{a}_p(i)$ from the corresponding true value $a_p(i)$. RE is a measure of overall mean square error and ACV is a measure of overall STD from the true values. Letting $\sigma(\hat{a}_p(i))$ be STD of estimates $\hat{a}_p(i)$ from its sample mean, we can watch bias by mental arithmetic of $|\sigma_t(\hat{a}_p(i)) - \sigma(\hat{a}_p(i))|$ preventing any misleading small $\sigma(\hat{a}_p(i))$. The exact CRB for each AR parameter [12] is also computed for reference. We mention that 1% and 20% of variance of $y(n)$ are used as the two variances for the artificial noise in the joint method regardless of noise type and SNR.

Fig. 3 shows the simulation results for the nonstationary colored noise case with linear SNR change. As we see in Fig. 3(a) and (b), the noise perturbation makes the noise poles more scattered. The final reconstructed AR spectrum in Fig. 3(c) illustrates almost perfect reconstruction.

For the purpose of comparison, we include two classic methods, the Yule-Walker (YW) method and the Modified YW (MYW) method as well as the methods in [3] and [6]. The YW method simply ignores any observation noise. The MYW method estimates the AR parameters by solving modified YW equations [11] which do not contain $r_y(0)$.

Since the last three methods were originally developed for stationary white noise with known model order, we first compare the joint method with them at different fixed SNRs as shown in Table I. To save space, Table I gives only first AR parameter estimate $\hat{a}_p(1)\pm\sigma(\hat{a}_p(1))$, $\pm\sigma_t(\hat{a}_p(1))$ along with RE and ACV. As expected, the YW method does not produce any reliable result even at 25dB SNR. The method in [3] seems to work at 20dB or 15dB, which outperforms the MYW method. The method in [6] is noteworthy because it robustly estimates the AR parameters in a wide range of SNR. The last column for the joint method shows a fair measure of noise immunity. To be fair, [6] is slightly better than the joint method at 20~30dB and, the joint method is slightly better at 15dB but substantially better under 10dB of SNR.

In the presence of nonstationary white noise, only [6] and the joint method work in practice. It is observed that [6] is more affected by abrupt change than linear change, but the joint method is not shaken by noise fluctuations and much superior to [6]. Table II shows the simulation results for the abrupt change. We omit the result for linear change because of space. Not surprisingly, all four other methods in Table I do not work at all and the method in [8] may operate with stationary 20dB colored noise. So we present performance of the joint method for 5dB stationary colored noise and two nonstationary colored noises in Table III, in which we see no performance degradation as in the previous two tables. Actually all these results are expected because we are making use of the nonstationarity of noise. Note that the joint method virtually achieves the CRB.

We wrap up this section by mentioning several strong points of the joint method. It provides AR parameter estimates along with the order. It is multifaceted in terms of the observation noise. It does not require any estimation of noise statistics since it does not use them analytically. The distance from the CRB is shaken a little (but not much) by the noise type. The price we should pay for these nice properties may be the multiple test-runs (around 10 here).

5. CONCLUDING REMARKS

A joint AR order and parameter estimation algorithm, which operates in general noise environment, is proposed. Two major contributions may be mentioned. First, the joint

TABLE I
INFLUENCE OF SNR ON THE PERFORMANCE
(STATIONARY WHITE NOISE, 1000 INDEPENDENT SIMULATIONS, ONLY THE ESTIMATES OF $a_p(1) = -2.7377$ ARE SHOWN,)

SNR		YW	MYW	[3]	[6]	Joint method
25dB	$\hat{a}_p(1)$	−1.8204	−2.7470	−2.7631	−2.7332	-2.7633
	σ	±0.0267	±0.1171	±0.0514	±0.0124	±0.0186
	σ_t	±0.9178	±0.1175	±0.0573	±0.0132	±0.0316
	RE	59.04 %	0.54 %	1.26 %	0.37 %	2.56 %
	ACV	342.25 %	8.01 %	3.25 %	0.98 %	3.11 %
15dB	$\hat{a}_p(1)$	−1.0789	−3.2978	−2.2966	−2.7348	−2.7409
	σ	±0.0199	±20.7964	±9.4272	±0.0169	±0.0102
	σ_t	±1.6590	±20.8039	±9.4375	±0.0172	±0.0106
	RE	90.25 %	34.49 %	21.88 %	0.29 %	0.19 %
	ACV	770.79 %	1040.98 %	616.31 %	1.31 %	0.71 %
5dB	$\hat{a}_p(1)$	−0.4850	−1.4010	−0.7896	−2.8094	−2.7248
	σ	±0.0203	±7.3727	±2.9878	±0.2238	±0.0046
	σ_t	±2.2528	±7.4929	±3.5668	±0.2350	±0.0137
	RE	97.52 %	83.79 %	96.28 %	5.04 %	1.64 %
	ACV	4397.15 %	3609.38 %	3030.68 %	23.00 %	2.02 %

TABLE II
ACCURACY COMPARISON FOR NONSTATIONARY WHITE NOISE
(***ABRUPT*** SNR CHANGE 25dB TO 5dB, 1000 INDEPENDENT SIMULATIONS)

		$\hat{a}_p(1)$	$\hat{a}_p(2)$	$\hat{a}_p(3)$	$\hat{a}_p(4)$
True Value		−2.7377	3.7476	−2.6293	0.9224
CRB		±0.0079	±0.0136	±0.0115	±0.0042
[6]		−2.7835	3.8511	−2.7272	0.9603
	σ	±0.1160	±0.2486	±0.2611	±0.1426
	σ_t	±0.1248	±0.2693	±0.2788	±0.1475
	RE	2.85 %			
	ACV	9.27 %			
Joint Method		−2.7385	3.7330	−2.6046	0.9069
	σ	±0.0049	±0.0106	±0.0104	±0.0045
	σ_t	±0.0049	±0.0181	±0.0268	±0.0161
	RE	0.60 %			
	ACV	0.87 %			

TABLE III
PERFORMANCE OF JOINT METHOD FOR COLORED NOISE
(1000 INDEPENDENT SIMULATIONS)

		$\hat{a}_p(1)$	$\hat{a}_p(2)$	$\hat{a}_p(3)$	$\hat{a}_p(4)$
True Value		−2.7377	3.7476	−2.6293	0.9224
CRB		±0.0079	±0.0136	±0.0115	±0.0042
Fixed 5dB SNR		−2.7419	3.7438	−2.6162	0.9126
	σ	±0.0057	±0.0111	±0.0107	±0.0047
	σ_t	±0.0071	±0.0117	±0.0169	±0.0108
	RE	0.32 %			
	ACV	0.60 %			
Linear Change		−2.7429	3.7536	−2.6295	0.9199
	σ	±0.0051	±0.0106	±0.0104	±0.0048
	σ_t	±0.0073	±0.0122	±0.0104	±0.0054
	RE	0.15 %			
	ACV	0.39 %			
Abrupt Change		−2.7443	3.7547	−2.6292	0.9192
	σ	±0.0050	±0.0102	±0.0101	±0.0046
	σ_t	±0.0082	±0.0125	±0.0101	±0.0055
	RE	0.19 %			
	ACV	0.40 %			

method is not affected by noise statistics and accordingly it does not require them. Especially and importantly, the AR parameter estimates nearly achieve the CRB in practice and are unbiased whether the observation noise is white or not and stationary or not. Secondly, the AR order is accurately obtained without any separate effort such as referring to any order criterion

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