FAST AND APPROXIMATIVE ESTIMATION OF CONTINUOUS-TIME STOCHASTIC SIGNALS FROM DISCRETE-TIME DATA

Magnus Mossberg

Karlstad University
Department of Electrical Engineering
SE-651 88 Karlstad, Sweden

Erik K. Larsson

Uppsala University
Department of Systems and Control
P.O. Box 337, SE-751 05 Uppsala, Sweden

ABSTRACT

A fast and approximative method for estimating continuous-time stochastic disturbance signals, described as continuous-time autoregressive moving average processes, from discrete-time data is presented. First, it is shown how these processes can be regarded as continuous-time autoregressive processes and the relation between the two types of processes is derived. The relation is then used for mapping estimated autoregressive parameters from an instrumental variable approach onto autoregressive moving average parameters. The procedure provides a solution to the estimation problem that preserves the continuous-time parameterization.

1. INTRODUCTION

Continuous-time stochastic system descriptions are important mathematical tools for modeling stochastic signals and stochastic disturbances in continuous-time, see, e.g., [1–5]. In this paper, continuous-time autoregressive moving average (CARMA) processes are studied with special interest. A CARMA process is defined as

$$A(p)y(t) = C(p)e(t), \tag{1}$$

where

$$A(p) = \sum_{i=0}^{n} a_i p^{n-i}, \tag{2}$$

$$C(p) = \sum_{j=0}^{m} c_j p^{m-j}, \tag{3}$$

where $n > m$, $A(p)$ has all zeros in the left half plane, p denotes the differentiation operator with respect to time, and

$$\mathrm{E}\{e(t)e(\tau)\} = \lambda^2 \delta(t-\tau). \tag{4}$$

Without loss of generality it is assumed that $a_0 = 1$. An important interpretation of the CARMA process is that it can be thought of as the underlying process for the spectrum

$$\phi(\mathrm{i}\omega) = \lambda^2 \frac{|C(\mathrm{i}\omega)|^2}{|A(\mathrm{i}\omega)|^2}. \tag{5}$$

The main purpose of the paper is to find a fast and reliable estimator for the parameters $\{a_i\}_{i=1}^{n}$ and $\{c_j\}_{j=0}^{m}$, based on discrete-time data $\{y(\ell h)\}_{\ell=1}^{N}$, where h denotes the sampling interval. Another important reason for studying this problem is that its solution is also the solution to the problem of estimating continuous-time AR (CAR) process parameters from discrete-time data corrupted by discrete-time measurement noise.

One possibility is to transform the CARMA process by instantaneous and exact sampling, see [6, 7], into a discrete-time ARMA process, estimate the discrete-time parameters and then transform them into continuous-time. The discrete-time parameters can be estimated using, e.g., the prediction error method, see [8], which yields consistent and statistically efficient estimates. This *indirect* approach is considered in [9], where also the inherent difficulties of estimating CARMA parameters are discussed.

When it comes to the mapping between the discrete-time and the continuous-time parameters, the relation between the poles is well-known through the exponential function. There is, however, no closed form expression for the relation between the zeros, see [9] and the references therein. Another possible drawback with an indirect approach using the prediction error method is that a nonlinear minimization problem has to be solved.

An interesting question is if something can be gained by keeping the continuous-time parameterization, i.e., by using a *direct* approach. A simple direct approach is to replace the differentiation operator in (1) with the delta operator $\delta = (q-1)/h$, where q denotes the forward shift operator, form a linear regression model and estimate the parameters with the least squares method, see [10]. However, due to the special structure of the CARMA process that includes unknown derivatives of $e(t)$, this approach is not possible to apply. On the other hand, this approach is successful for estimating the parameters in CAR processes, see [10–12]. An idea is therefore to investigate if it is possible to rewrite a CARMA process as a CAR process. If so, the CAR parameters could be estimated as in [10–12] and then be transformed into the searched CARMA parameters. In most cases, this approach is expected to be less accurate, but considerably faster, than the indirect approach using the prediction error method. The estimates given by the approach proposed here can also be used as initial values in the nonlinear minimization problem that has to be solved when using the prediction error method.

2. PROCESS APPROXIMATION

It is well-known that an ARMA process can be written as an AR process of infinite order, and depending on the locations of the poles and zeros of the ARMA process, a reasonable approximation can in practice be given by a finite order AR process, [13]. Here, this is done for the corresponding continuous-time case, i.e., the

CARMA process (1) is approximated by the CAR process

$$\Lambda(p)y(t) = e(t) \tag{6}$$

of order $\eta > n$, where

$$\Lambda(p) = \sum_{i=0}^{\eta} \lambda_i p^{\eta-i} \tag{7}$$

with $\lambda_0 = 1$.

First, (1) is written on the form

$$A(p)y(t) = \sum_{j=0}^{m} c_j p^{m-j} e(t) = \Big(\prod_{j=1}^{m}(d_j p + 1)\Big)e(t), \tag{8}$$

where it is assumed that $c_m = 1$ and $|d_j| < 1, \forall j$. It follows that

$$pA(p)y(t) = \sum_{j=0}^{m} c_j p^{m-j+1} e(t). \tag{9}$$

Now, (9) is solved with respect to $pe(t)$ and the result is used in (8) which gives

$$\begin{aligned}(1 - c_{m-1}p)A(p)y(t) &= (1 - c_{m-1}p)\sum_{j=0}^{m-2} c_j p^{m-j} e(t) \\ &\quad + c_{m-1}^2 p^2 e(t) + e(t).\end{aligned} \tag{10}$$

The CAR part in the left hand side of (10) is of order $n+1$, whereas the continuous-time MA (CMA) part in the right hand side can be expressed as

$$f_{m+1}p^{m+1}e(t) + \ldots + f_2 p^2 e(t) + e(t), \tag{11}$$

where $f_2, \ldots, f_{m+1}$ are functions of $\{c_j\}_{j=0}^{m-1}$. Moreover, f_i, $i = 2, \ldots, m+1$, consists of sums of $\mathcal{O}(d_1 d_2 \cdots d_i)$ terms.

The above procedure is now repeated, i.e., both sides of (8) are differentiated twice, the resulting description is solved for $p^2e(t)$, and the result is substituted back into (10). The CAR part of the resulting process is of order $n+2$, whereas the CMA part can be written as

$$g_{m+2}p^{m+2}e(t) + \ldots + g_3 p^3 e(t) + e(t), \tag{12}$$

where g_i, $i = 3, \ldots, m+2$, is built of sums of $\mathcal{O}(d_1 d_2 \cdots d_i)$ terms.

If the procedure is repeated k times, the CAR part of the resulting process is of order $n+k$ and the CMA part has the form

$$\gamma_{m+k}p^{m+k}e(t) + \ldots + \gamma_{1+k}p^{1+k}e(t) + e(t), \tag{13}$$

where the function γ_i, $i = 1+k, \ldots, m+k$, is built of sums of $\mathcal{O}(d_1 d_2 \cdots d_i)$ terms. Since $|d_j| < 1$, $\forall j$, (13) can be approximated by $e(t)$ for rather moderate values of k, which means that the resulting process can be regarded as a CAR process of order $n+k$. The smaller values of $\{|d_j|\}_{j=1}^{m}$, the smaller k is needed, and the restriction $|d_j| < 1, \forall j$, simply means that the zeros of (8) are located outside the region $[-1, 1]$.

2.1. Second order case

The procedure of finding an approximation to a CARMA process in form of a CAR process is exemplified for the second order case,

$$(p^2 + a_1 p + a_2)y(t) = (dp + 1)e(t), \tag{14}$$

where $|d| < 1$. It follows that

$$(p^3 + a_1 p^2 + a_2 p)y(t) = (dp^2 + p)e(t). \tag{15}$$

Solve this equation for $pe(t)$ and substitute the result back into (14) to get

$$\begin{aligned}&\big(-dp^3 + (1 - da_1)p^2 + (a_1 - da_2)p + a_2\big)y(t) \\ &\quad = (1 - d^2p^2)e(t).\end{aligned} \tag{16}$$

Except for the term $-d^2p^2e(t)$, this looks like a CAR(3) process. Now, repeat the above procedure, i.e., write (14) as

$$(p^4 + a_1 p^3 + a_2 p^2)y(t) = (dp^3 + p^2)e(t), \tag{17}$$

solve for $p^2e(t)$ and substitute the result back into (16) to get

$$\begin{aligned}&\big(d^2p^4 + (d^2a_1 - d)p^3 + (1 + d^2a_2 - da_1)p^2 \\ &\quad + (a_1 - da_2)p + a_2\big)y(t) = (1 + d^3p^3)e(t).\end{aligned} \tag{18}$$

This resembles a CAR(4) process, except for the d^3p^3 term. If the procedure is repeated k times, the process

$$\begin{aligned}&\big((-d)^k p^{2+k} + \alpha_1 p^{1+k} + \ldots + \alpha_{2+k}\big)y(t) \\ &\quad = (1 + (-1)^k d^{1+k} p^{1+k})e(t)\end{aligned} \tag{19}$$

is obtained. Since $|d| < 1$, the right hand side can be approximated by $e(t)$, which means that the process can be regarded as a CAR process of order $2+k$.

The spectrum of the CARMA process (14) is shown in Fig. 1 together with the spectra for the approximate CAR processes given by (19), with the right hand side approximated by $e(t)$, for $k = 1$, 2 and 4, for $a_1 = 2$, $a_2 = 2$ and $d = 0.05$. The spectra are plotted for frequencies less than the Nyquist frequency when using the sampling interval $h = 0.01$ s for sampling the continuous-time processes. It is evident that the true CARMA spectrum is well described by the CAR spectra for frequencies up to the break frequency $\omega = 20$ rad/s, and that the best description is given by the spectrum corresponding to $k = 4$. However, for higher frequencies, the best description is given by the spectrum corresponding to $k = 1$.

3. CAR PARAMETER ESTIMATION

After approximating the CARMA process (1) with the CAR process (7), the CAR parameters $\{\lambda_i\}_{i=1}^{\eta}$ are to be estimated from the discrete-time data $\{y(\ell h)\}_{\ell=1}^{N}$. One possibility of estimating the parameters is to use the asymptotically ($N \to \infty$ and $h \to 0$) consistent instrumental variable approach described in [12]. There, the rth order differentiation operator p^r is approximated by the difference operator

$$D^r = \frac{1}{h^r}\sum_s \kappa_{r,s} q^s, \tag{20}$$

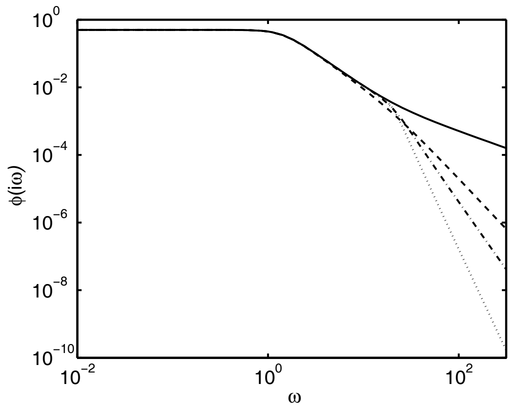


Fig. 1. The spectrum for (14) (solid) and the spectra for (19), with the right hand side approximated by $e(t)$, for $k = 1$ (dashed), 2 (dash-dotted) and 4 (dotted), for $a_1 = 2$, $a_2 = 2$ and $d = 0.05$.

where q is the forward shift operator. In order to get an exact differentiation as $h \to 0$, the weights $\{\kappa_{r,s}\}$ have to fulfill certain conditions, see [12]. Then, by introducing the regression vector and the instrumental variables

$$\boldsymbol{\varphi}^T(\ell h) = \begin{bmatrix} -D^{\eta-1}y(\ell h), & \ldots & , -y(\ell h) \end{bmatrix} \tag{21}$$

and the instrumental variables

$$\boldsymbol{z}(\ell h) = \begin{bmatrix} y(\ell h), & \ldots & , y(\ell h - \eta h + h) \end{bmatrix}^T, \tag{22}$$

the estimates $\{\hat{\lambda}_i\}_{i=1}^{\eta}$ are found as

$$\begin{bmatrix} \hat{\lambda}_1 \\ \vdots \\ \hat{\lambda}_\eta \end{bmatrix} = \left(\sum_{\ell=1}^{N} \boldsymbol{z}(\ell h) \boldsymbol{\varphi}^T(\ell h) \right)^{-1} \left(\sum_{\ell=1}^{N} \boldsymbol{z}(\ell h) D^\eta y(\ell h) \right). \tag{23}$$

4. PARAMETER MAPPING

Once the CAR parameters $\{\lambda_i\}_{i=1}^{\eta}$ in (6) are estimated, the corresponding CARMA parameters $\{a_i\}_{i=1}^{n}$ and $\{c_j\}_{j=0}^{m}$ in (1), or $\{a_i\}_{i=1}^{n}$ and $\{d_j\}_{j=1}^{m}$ in (8), can be given by a suitable mapping function $\mathcal{F} : R^\eta \mapsto R^{n+m}$. It is nontrivial to find an expression for $\mathcal{F}$ that holds for all η, n and m. Therefore, the strategy for finding $\mathcal{F}$, which is the same for all η, n and m, is illustrated next for the second order case.

4.1. Second order case

Assume that the CARMA process (14) is written in the form (19), and that k is sufficiently large so that it is possible to consider (19) as the CAR($2 + k$) process

$$(p^{2+k} + \beta_1 p^{1+k} + \ldots + \beta_{2+k}) y(t) = v(t), \tag{24}$$

where $\beta_i = \alpha_i/(-d)^k$ and $v(t) = e(t)/(-d)^k$. Further, assume that its parameters $\beta_1, \ldots, \beta_{2+k}$ are estimated by using the instrumental variable method described in Section 3. The estimated CAR parameters will depend on the unknown and searched CARMA parameters a_1, a_2 and d, giving the system of equations

$$\boldsymbol{\Phi\theta} = \boldsymbol{\mu}, \tag{25}$$

where

$$\boldsymbol{\Phi}_{i,1} = (-1)^{i+1} d^{k+1-i}, \qquad i = 1, \ldots, k+1 \tag{26}$$

$$\boldsymbol{\Phi}_{j,2} = (-1)^{j} d^{k+2-j}, \qquad j = 2, \ldots, k+2 \tag{27}$$

$$\boldsymbol{\Phi}_{1,2} = \boldsymbol{\Phi}_{k+2,1} = 0, \tag{28}$$

$$\boldsymbol{\theta} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}^T, \tag{29}$$

$$\boldsymbol{\mu}_{l,1} = \begin{cases} d^k \beta_l + (-1)^{l+1} d^{k-l}, & l = 1, \ldots, k \\ d^k \beta_l, & l = k+1, k+2 \end{cases} \tag{30}$$

The system of equations (25) is linear in the CAR parameters a_1 and a_2 but nonlinear in the CMA parameter d, and the estimation problem can therefore be regarded as a separable least squares problem, [14]. This does not hold only for the second order case, but for all η, n and m. As a first step, consider the loss function

$$V(\boldsymbol{\theta}, d) = \|\boldsymbol{\mu} - \boldsymbol{\Phi\theta}\|^2 \tag{31}$$

from which an estimate of $\boldsymbol{\theta}$ as a function of d is obtained as

$$\hat{\boldsymbol{\theta}}(d) = \boldsymbol{\Phi}^\dagger \boldsymbol{\mu}, \tag{32}$$

where $\boldsymbol{\Phi}^\dagger$ denotes the pseudo-inverse of $\boldsymbol{\Phi}$. Then, as a second step, the estimate (32) is inserted into (31) to get the concentrated cost function

$$J(d) = V(\hat{\boldsymbol{\theta}}(d), d) = \|(\boldsymbol{I} - \Phi\Phi^\dagger)\boldsymbol{\mu}\|^2 \tag{33}$$

from which an estimate $\hat{d}$ is obtained as

$$\hat{d} = \arg\min_d J(d). \tag{34}$$

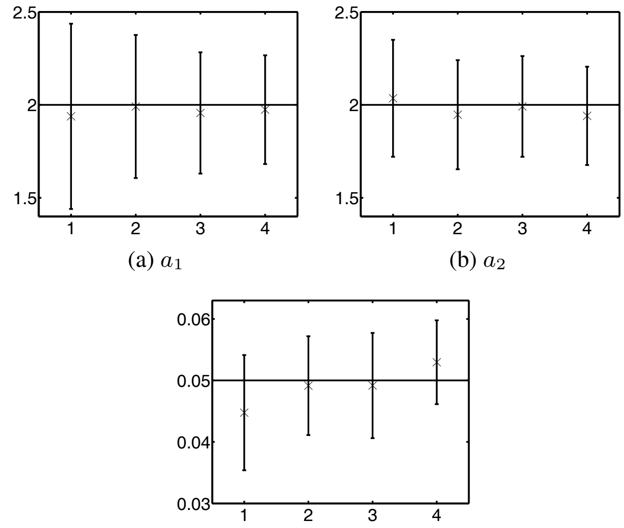
Finally, an estimate $\hat{\boldsymbol{\theta}}$ can be given from (32) and (34).

5. NUMERICAL STUDIES

Data is generated by instantaneous and exact sampling, see [6], of the second order CARMA process (14) where $a_1 = 2$, $a_2 = 2$ and $d = 0.05$, with sampling interval $h = 0.01$. The noise intensity λ^2 in (4) is chosen such that $y(t)$ gets unit variance. The parameters $\beta_1, \ldots, \beta_{2+k}$ in the CAR($2 + k$) process (24) are estimated from the data using $N = 10\,000$ data points for $k = 1, \ldots, 4$ using the instrumental variables technique described in Section 3, with the difference operator (20) chosen as the delta operator. The estimated CAR parameters are then mapped onto the CARMA parameters as described in Section 4.1. The whole procedure is repeated 200 times in a Monte Carlo simulation. It should be mentioned that some obviously erroneous estimates that occurred during this simulation study were removed.

The results are illustrated in Fig. 2 and a number of observations can be made. For the estimate of a_1, the variance is decreasing when k is increasing. This is not the case for the estimate of a_2, where the variance is about the same, regardless of the value of k. On the other hand, the variance for the estimate of a_2 is lower than the variance for the estimate of a_1, for all considered values of k. The biases for the estimates of a_1 and a_2 are about the same and do not change considerably with k. For the estimate of d, the variance is slightly decreasing when k is increasing, whereas the bias is smallest for $k = 2, 3$.

A possible explanation to these results can be given by observing Fig. 1, where it is seen that the CAR process corresponding



(a) a_1 (b) a_2 (c) d

Fig. 2. The mean values and the empirical standard deviations for the estimates of (a) a_1, (b) a_2 and (c) d as functions of k.

to $k = 1$ gives a better description of the CARMA spectrum for higher frequencies than the CAR process corresponding to $k = 4$. It is of course important that the CAR($2 + k$) process gives a good description of the CARMA spectrum for all frequencies. This suggests that a low pass filtering of the data with a cut-off frequency $\omega \approx 20$ rad/s might be fruitful.

Another explanation to the results in Fig. 2 is that the inaccuracy of the estimated CAR parameters in terms of bias and variance are mapped onto the searched CARMA parameters. Then the question is if it is always better to estimate more CAR parameters, since more estimated CAR parameters also mean a larger total variance.

6. CONCLUSIONS

A fast and approximative method for estimating continuous-time stochastic signals in forms of CARMA processes from discrete-time data was presented. The CARMA process was rewritten as a CAR process whose parameters were estimated using an instrumental variable technique, and then mapped onto the searched CARMA parameters.

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