SEMIPARAMETRIC SKEW-SYMMETRIC MODELING OF PLANAR SHAPES

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ABSTRACT

Shape modeling and template learning form an important area of research in image analysis. This paper addresses the problem from a novel viewpoint – using a new class of semiparametric skew distributions.

Given several realizations of a shape, we represent its template as a joint distribution of angle and distance from the centroid for all points on the boundary. The shape boundary may be arbitrary and irregular but simple. Its corresponding distribution is learned from the scattered data points of the available boundary realizations. We first obtained a bimodal distribution of the radii distances for given angles and subsequently synthesize the overall joint distributions according to some prior on the angles. We substantiate our proposed methodology with a number of examples.

1. INTRODUCTION

The goal of shape modeling is to seek mathematical representations to capture the intrinsic morphologies of various shapes and to account for their variability. Formally part of *pattern theory*, whose formalism is to a large extent due to Grenander [1], it seeks to quantify the structure of patterns present in an image. In recent years, the problem has been approached in numerous ways including rigid models [2] as well as flexible models [1, 3, 4, 5, 6]. While rigid models have been popular for many applications, their inability to reflect the inherent variability of shapes (e.g., anatomical shapes) has led to other more flexible approaches such as intersite arc distributions [1] and active shape models [3], which are deterministic/probabilistic hybrid models.

In contrast to all above approaches, we view the variability of shape as one that allows realization contours to remain within a certain neighborhood range around the mean. This in turn suggests that for any given angle around a shape, a probability density function may be found to capture the corresponding potential excursion of the curves at the given angle.

We specifically exploit a class of semiparametric skew symmetric distributions [7, 8, 9] due to potential skewness of data which may arise in practical problems. Simulations show that the method works equally well for non-skewed data.

The paper consists of a problem statement section given next. In Section 3, we describe the probabilistic model we wish to develop. In Section 4, we provide illustrating examples and finally some conclusions in Section 5.

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2. PROBLEM STATEMENT

Let a shape S_i be given by a curve $C_{S_i}(t)$

$$C_{S_i} : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R} \times [0, \pi] \quad (1)$$

and for convenience and clarity we take $I \subset N$ (a sampled curve). Given a set $\{{C_{S_i}}^j\}_{j=1,\ldots,N}$, we ask to provide a probabilistic model for S_i in terms of its radius (from the centroid) and angle around. Note that alternatively we may view $C_{S_i}(t)$ as a parametric representation

$$(x(t), y(t)) \quad \text{or} \quad \left(\sqrt{x^2(t) + y^2(t)}, \arctan(y(t)/x(t))\right).$$

Given several realizations of a shape, learning is equivalent to capturing all objects having closely similar shape realizations. Note that heart shapes, for instance, when given in two images, may differ due to occlusion, noise, clutter or difference in pose. Despite this variability, it is reasonable to assume that these different realizations share much in common, and hence be able to model all of the deviations and capture the essence of the shape.

When normalized to a pre-specified area, the boundaries of these realizations will lie within some standard neighborhood as illustrated in Fig. 1(b). Combining all such realizations is tantamount to assembling a cloud of points in the neighborhood of a template boundary as shown in Fig. 1(c). By sampling the realizations at specified angles, the points in the cloud may be assumed iid or at most may be modeled as a first order Markov process. The boundary of any realization of a given shape will be a subset of points within a tubular cloud, interpreted as a permissible shape domain, as shown in Fig. 1(c).

$$S_i \sim \{(\sqrt{x^2 + y^2}_i + n_{r_i}, \arctan(y/x)_i + n_{\theta_i})\} \quad (2)$$

where n_{r_i} and n_{θ_i} represent variability within the admissible domain.

Based on these realizations, the points in the cloud at a given angle may be shown to be distributed around the template boundary according to a skewed distribution $p(r, \theta)$. Note that the non-skewed densities are a particular case of skewed representation.

3. SHAPE ANALYSIS

As mentioned earlier, in this paper, we will be investigating a class of closed shapes in the same view as those in Fig. 3.

For convenience, we adopt a polar coordinate system (r, θ). In addition, we choose to translate the origin to the centroid of the shape.

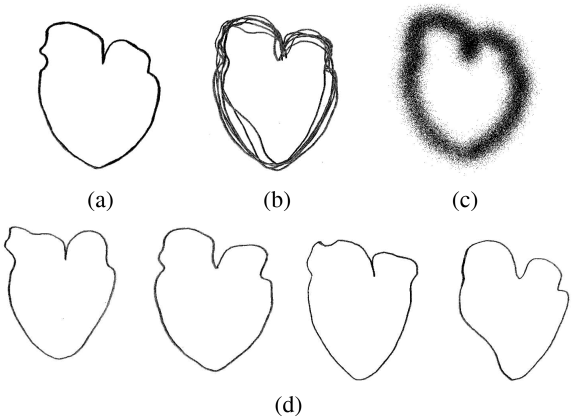

Fig. 1. (a) Heart shape; (b) Some realizations superimposed on each other; (c) Sampled superimposed realizations; (d) Constituent realizations.

Following the determination of a centroid for a given shape, we proceed to randomly sample it at angles $\theta \in [0, \pi]$ according to a prior distribution $p(\theta)$. For a given fixed $\theta = \theta_I$, we identify all samples lying within an ϵ-neighborhood of θ_I. In light of the nature of the shape, we can associate two clusters of samples on either side of the centroid at θ_I, with a relative phase difference of π. The two clusters are distributed according to a bimodal conditional distribution (for fixed θ_I), $p(r|\theta_I)$. We propose the following model to represent such a class of conditional distributions:

$$p(r|\theta) = 2\omega f\left(\frac{r-\xi}{\sigma}\right) H\left(P_K\left(\frac{r-\xi}{\sigma}\right)\right) \tag{3}$$

where f is any symmetric pdf and H is any cdf of a continuous random variable that is symmetric around zero and P_K is K-order polynomial. ξ and σ are respectively the location and scale parameters. ω is a parameter that makes $p(r|\theta)$ a valid density. One can show that such a formulation affords much flexibility such as multimodality, skewness, symmetry, etc. [10].

Upon specifying the conditional distribution model, and using a data sample of sufficient size, we may proceed to learn the density by standard techniques, for instance, maximization of the log-likelihood function:

$$\begin{aligned} L(\xi, \sigma, \boldsymbol{\alpha}) &= m \log\left(2\omega(\xi, \sigma, \boldsymbol{\alpha})\right) + \sum_{i=1}^{m} \log f\left(\frac{r_i - \xi}{\sigma}\right) \\ &+ \sum_{i=1}^{m} \log H\left(P_K\left(\frac{r_i-\xi}{\sigma}\right)\right), \end{aligned} \tag{4}$$

where m is the number of radius samples, $r_1, \ldots, r_m$, while the unknown parameters are the location parameter ξ, polynomial coefficients $\boldsymbol{\alpha}$, and the standard deviation σ.

3.1. Prior Distribution

The challenge in determining a prior for the angle in a shape descriptor is the presence of singularities such as shown as a cusp in Fig. 1(a). Given the presence of such events and their nonuniform occurrence throughout, suggests a piecewise uniform or piecewise tapered uniform distribution for the prior as shown in Fig. 2.

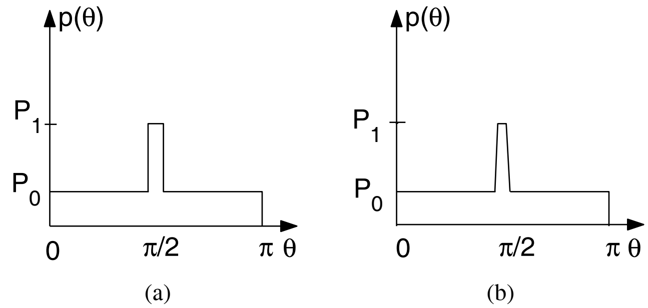

Fig. 2. Prior distribution: (a) Piecewise uniform; (b) Piecewise tapered.

3.2. Template Learning

With a prior $p(\theta)$ in hand together with a conditional density for r, $p(r|\theta)$, we are in a position to construct the overall density for a shape. If we discretize the angle space, the overall shape may be represented by the following joint density $p(r, \boldsymbol{\theta})$:

$$p(r, \boldsymbol{\theta}) = \prod_{\theta \in [0,\pi]} p(r|\theta) p(\theta) \tag{5}$$

Note that the nature of the distribution will dictate the parameters of importance for template extraction, e.g., a mode may be more appropriate for a skewed density. Hence, corresponding to each angle, we estimate the modes of the posterior, which are assumed to lie on the boundary. In the limiting case, where the number of angle samples goes to infinity, the set of modes will constitute the closed contour of our template.

3.3. Prior performance assessment

In order to assess the quality of a prior selection, we evaluate the cumulative deviation between an "*ideal*"[1] shape and learned template with specific priors.

Before presenting the performance measure, we discuss the relation between realizations and template. Let us denote the two modes at a specified angle, θ, by $\hat{r}_1(\theta)$ and $\hat{r}_2(\theta)$, which represent the template boundary. For each realization from the data set, corresponding to the same angle θ, the deviation of the boundary points $r_1(\theta)$ and $r_2(\theta)$ from the corresponding modes is given by:

$$dr_i(\theta) = r_i(\theta) - \hat{r}_i(\theta). \tag{6}$$

Hence, we can generate any realization in the permissible domain, the tubular region, by adding the deviations $dr_i(\theta)$ to the template corresponding to each angle.

As a performance measure, we may use the L_2-norm of the difference between the two shapes as defined below:

$$D = \sqrt{\int_0^{\pi} \left(dr_1^2(\theta) + dr_2^2(\theta)\right) d\theta}. \tag{7}$$

Discretization of angle space and considering some ϵ-neighborhood of θ yield the form given by Eq. 8 for the departure of learned tem-

[1] Ideal shape is the realization of a shape as viewed from a perfect angle in noise-free environment.

plate from the ideal shape.

$$D = \sqrt{\sum_{\theta \in [0,\pi]} \sum_{\theta_i \in N_\epsilon^\theta} \left(dr_{1,\theta_i}^2 + dr_{2,\theta_i}^2 \right)}, \tag{8}$$

where N_ϵ^θ is some ϵ-neighborhood around θ and dr_{i,θ_j} is the deviation for the i-th mode at a given angle θ_j in N_ϵ^θ.

4. EXPERIMENTAL RESULTS

In this section, we give some practical applications of the proposed technique. We have tried to give diverse examples related to various fields of applications instead of only focussing on medical applications. These results demonstrate the generality and effectiveness of the proposed method. Instead of working with synthetic images, we have concentrated more on real images as our test cases. Later in the section, we describe how to simulate different realizations using the model of Eq. 5.

We learn templates for three shapes acquired from real images identified as heart, brain, and star which are shown in Fig. 3, using different priors. The templates learned with uniform prior are given in Fig. 4, while those with piecewise tapered uniform prior are illustrated in Fig. 5. Note that we took 20 angle samples in each case except for the Star, which is learned with only 10 samples. In addition, we used third order polynomial with normal distribution in Eq. 3. Templates learned with 100 samples are shown in Fig. 6. The performance measure D for the three cases is tabulated in Table 1. It is obvious both visually and from the performance measures that piecewise uniform prior gives better results than uniform prior. In particular, in the case of the heart, the performance measure is approximately the same for both priors. This is the price that we pay, when we try to weight certain directions more than others while using small sample size and hence end up degrading a shape in other directions.

Table 1. PERFORMANCE MEASURE D FOR THE THREE CASE STUDIES

Case Study	Prior for angle θ	D ($\times 10^3$)
Star	Uniform	2.1
	Piecewise uniform tapered	1.6
Brain	Uniform	0.5
	Piecewise uniform tapered	0.4
Heart	Uniform	0.205
	Piecewise uniform tapered	0.211

4.1. Sampling from models

Using the learned conditional distribution and the prior, we can generate all possible realizations of the shape. This is demonstrated by simulating a cloud of points according to the parameters learned for a circle with two singularities (Fig. 7). The simulated cloud is illustrated in Fig. 7(c), which clearly shows that the two clouds are similar.

5. CONCLUSIONS

In this paper, we discussed a novel method for template learning, which accounts for shape variations in different realizations, using semiparametric skew-symmetric distributions. It involves the discretization of angle space, and then learning posterior density for the radii given angle, chosen according to some prior. The shape may be recovered by capturing the local maxima (modes) of the posterior for all angle samples.

We presented several case studies related to different application areas using real images. Based on computer simulations, semiparametric skew-symmetric template learning appears to be quite effective and robust method for capturing the variability inherent to shapes. It can capture the shape singularities to some extent and may be applied to complex multi-loop templates using higher order polynomial P_K in Eq. 3. Since the joint distribution $p(r, \boldsymbol{\theta})$ given by Eq. 5 represents the entire class of shapes that lie in the permissible shape domain, the learned model can be used to simulate any realization of the shape, .

The method can also be extended to 3-D shapes, which we are currently investigating.

6. REFERENCES

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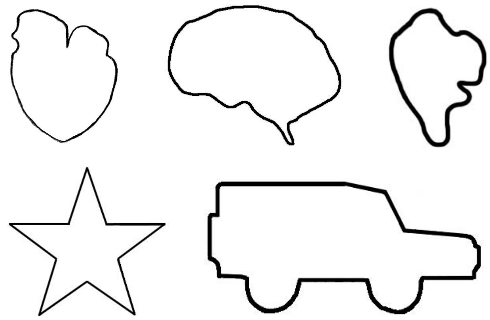
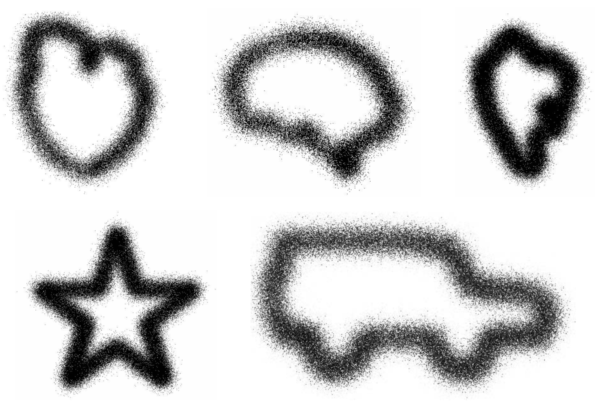
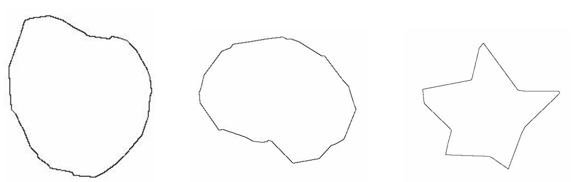


(a)



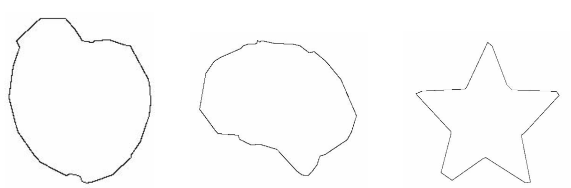
(b)

Fig. 3. (a) Actual shapes: heart, brain, tumor, star and car; (b) Corresponding realizations.



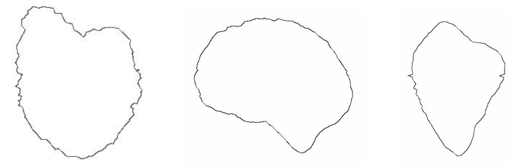
(a) (b) (c)

Fig. 4. Template learned using uniform prior: (a) Heart; (b) Brain; (c) Star.

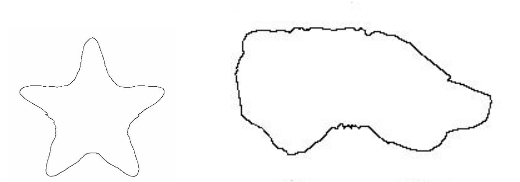


(a) (b) (c)

Fig. 5. Template learned using piecewise tapered uniform prior: (a) Heart; (b) Brain; (c) Star.

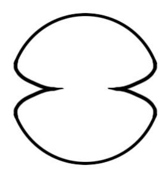


(a) (b) (c)

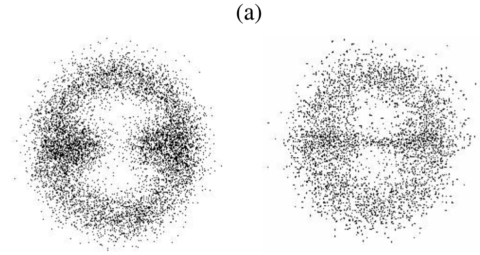


(d) (e)

Fig. 6. Template learned with 100 angle samples. (a) Heart; (b) Brain; (c) Tumor; (d) Star; (e) Car.



(a)



(b) (c)

Fig. 7. Shape simulation according to joint distribution of Eq. 5: (a) Ideal shape – a circular shape with two singularities; (b) Realizations; (c) Realizations simulated using parameters learned for (b).