SINGULAR ARMA SIGNALS

Bernard Picinbono † *and Jean-Yves Tourneret**

† École Supérieure d'Électricité, Plateau de Moulon, 3 rue Joliot–Curie 91192, Gif sur Yvette, France
* IRIT/ENSEEIHT/TéSA, 2 rue Charles Camichel, BP 7122, 31071 Toulouse cedex 7, France
bernard.picinbono@lss.supelec.fr, jean-yves.tourneret@tesa.prd.fr

ABSTRACT

Singular random signals are collections of singular random variables indexed over time. Singular random variables have continuous distribution with derivatives equal to zero almost everywhere. Such random variables do not seem interesting in Signal Processing. On the contrary, this paper shows that simple signals can be singular. This is especially the case of ARMA signals generated from discrete white noise with poles located inside a so-called singularity circle. The origin of singularity and its relations to a fractal structure are presented along with various simulations illustrating the theoretical results.

1. INTRODUCTION

Singular random variables (RV) are characterized by continuous distribution functions (DF) with derivatives equal to zero almost everywhere. Therefore, neither the probability density function (PDF) nor the probability mass exist at some points. Singular RVs are well known in probability theory and discussed in more mathematically oriented books (see [1, p. 180] or [2, p. 9]). Examples of singular RVs are also given in [3] which contains a long list of references.

Singular RVs are considered as mathematical curiosities in the Signal Processing community. Thus, they are either ignored [4], [5] or presented without any practical interest [6]. This idea is noted in [3, p. 4]: "In applications one almost inevitably encounters either discrete or absolutely continuous distributions. Singular distributions are interesting from a theoretical viewpoint but hardly ever occur in practical work."

However, [3] has shown that if w_k is a symmetric Bernoulli strictly white noise (SBWN) (a sequence of independent and identically distributed (IID) binary RVs with values ± 1 with probability $\frac{1}{2}$), then the RV $x = \sum_{k=0}^{\infty} a^k w_k$ is singular for $a < 1/2$. This is the simplest example of a singular RV which also appears often in Signal Theory. Indeed, this RV is the value at each time of an autoregressive signal of order one [AR(1)] generated from the simplest of white noises. Thus, singular RVs are actually quite common in Signal Processing. This has already been noted in [7].

This paper shows that the singularity of ARMA signals results from a combination of two points: discrete-valued inputs (very common in communication problems) and location of the poles inside the so-called *singular circle*. This circle is centered at the origin with a radius equal to $1/q$, where q is the number of discrete values of the input signal. The singular circle is the cornerstone of the singularity discussion, as is the unit circle for stability. The theory of the problem is introduced in Section 2. Various computer simulations are presented in Sections 3 and 4. This approach will highlight a phenomenon considered a mathematical curiosity and thus far widely ignored.

2. THEORETICAL APPROACHES

2.1. Singularity

According to Lebesgue's decomposition theorem, any DF F can be decomposed uniquely as [3]:

$$F(x) = a_1 F_d(x) + a_2 F_{ac}(x) + a_3 F_s(x), \tag{1}$$

where the coefficients a_i satisfy $\sum_{i=1}^{3} a_i = 1$, $a_i \geq 0$, and F_d, F_{ac}, F_s are three distribution functions such that:
• F_d is a step function, i.e. the DF of a *discrete* RV,
• F_{ac} is an *absolutely continuous* (AC) function, i.e. the DF of a continuous RV with a probability density function,
• F_s is a *singular* DF, i.e. a continuous function whose derivative is almost everywhere equal to zero. Consequently, there is neither PDF nor probability mass at some point.

A distribution F is said to be *pure* if one coefficient in (1) is equal to 1 (and the two others equal 0).

Consider a discrete white noise w_k which is a sequence of IID RVs taking q distinct possible values. The simplest example is the SBWN where $q = 2$. Denote x_k as the output of a causal filter with impulse response (IR) h_k driven by w_k:

$$x_k = \sum_{l \geq 0} h_l w_{k-l}. \tag{2}$$

This convolution is a finite sum if the IR of the filter is finite (FIR) or a series if it is infinite (IIR). In the latter case, it is assumed that the IR h_k is absolutely convergent such that

$$S = \sum_{k=0}^{\infty} |h_k| < \infty. \tag{3}$$

The discrete type of the input white noise implies the so-called *law of pure types* (see [3, p. 64]). The law of pure types states that the RV $x = \sum_{k=0}^{\infty} h_k w_k$ (the output at each time of the filter defined by (2)) is either purely AC, or purely discrete, or purely singular. If there are only n terms in the sum defining x (FIR), x is purely discrete and has q^n values at the maximum. It is assumed in what follows that the filter is IIR. This implies there is no N such that for $k > N$, $h_k = 0$. With this assumption, it can be shown that x cannot be purely discrete (for instance by using theorem 3.7.6 of [3, p. 63]). As a consequence, x is purely AC or purely singular.

Consider the case of the *exponential filter* with $h_k = a^k u_k$ (where $u_k = 1$ for $k \geq 0$ and $u_k = 0$ for $k < 0$). If w_k is a

SBWN, the following holds [3][8]: i) x is purely singular if $0 < a < 1/2$, x is uniformly distributed in the interval $[-2, +2]$ if $a = 1/2$ and iii) x is purely AC for almost all values of a satisfying $1/2 < a < 1$. Note that x is a singular RV for some $a \in]1/2, 1[$ (such as the reciprocals of the so-called Pisot numbers [8]).

Our aim is to extend this kind of result to more general filters and discrete white noises. Specifically, extensions to dynamical filters defined by their poles and zeros (ARMA signals) are studied. For this purpose, recall some notation introduced in [3]. The spectrum S_{F_x} of the DF $F_x(\xi)$ of the RV x is the set of all the points of variation (increase) of S_{F_x}. Denote $L(S_{F_x})$ as the Lebesgue measure of this set. The RV x cannot be purely AC if $L(S_{F_x}) = 0$. Thus, x is either purely discrete or purely singular. However, x cannot be purely discrete as indicated previously. Consequently, the *singularity* is characterized by the relation $L(S_{F_x}) = 0$. In order to study $L(S_{F_x})$, consider the quantity

$$\rho_n = \sum_{k=n}^{\infty} |h_k| \tag{4}$$

(the rest of the series giving S in (3)). The most common characterization of the singularity is:

Theorem 1 *[3, p. 66]. If w_k is a SBWN and if $h_n > 0$, $h_n > \rho_{n+1}$ for all $n \geq 0$, then*

$$L(S_{F_x}) = 4 \lim_{n \to \infty} 2^n h_n. \tag{5}$$

Consider some consequences of this theorem:

- Since $h_n = \rho_n - \rho_{n+1}$, the condition $h_n > \rho_{n+1}$ can also be written as $\rho_n > 2\rho_{n+1}$, $n \geq 0$.
- Since $\rho_0 = S$ and $\rho_1 = S - h_0$, the condition on ρ_n implies that $S > 2(S - h_0)$ or $S < 2h_0$. For the IR $h_k = a^k u_k$, $h_0 = 1$ and $S = 1/(1-a)$. Thus, the condition $S < 2h_0$ yields $a < 1/2$. It is easy to verify that this implies $\rho_n > 2\rho_{n+1}, \forall n > 0$. Therefore (5) can be used and, yields $L(S_{F_x}) = 0$ since $a < 1/2$. Consequently, the RV x is singular. This is the result indicated above. Furthermore, note that (5) is still valid for $a = 1/2$ since $L(S_{F_x}) = 4$ and the RV x is uniformly distributed in $[-2, +2]$ as indicated above.

However, the conditions allowing (5) are too restrictive. Instead of computing the exact value of $L(S_{F_x})$, it is more interesting to obtain an upper bound of this measure. Furthermore, it is convenient to have a result valid for any discrete-valued signal w_k. This is the purpose of the following proposition.

Theorem 2. *Let q be the number of distinct possible values of the discrete IID random variables w_k. If the filter with impulse response $g_k = q^k h_k$ is stable, then the random variable $x = \sum_{k=0}^{\infty} h_k w_k$ is singular.*

Proof. Let A be the greatest possible value of $|w_k|$. The partial sum $\sum_{k=0}^{n-1} h_k w_k$ has q^n distinct values v_i^n at the maximum. The possible values of the RV x are of the form $v_i^n + \sum_{k=n}^{\infty} h_k \eta_k$, where η_k is one of the possible values of w_k. These values belong to an interval I_i^n of measure smaller than $2A\rho_n$. Since there are at the maximum q^n intervals I_i^n, the possible values of x belong to an interval I_n of measure smaller than $2Aq^n\rho_n$. Outside this interval, the DF $F(\xi)$ of x cannot vary. Thus, the measure of the set of points of increase of $F(\xi)$ is smaller than $2Aq^n\rho_n$ for any $n \geq 0$. Consequently, the measure $L(S_{F_x})$ of the spectrum S_{F_x} satisfies $L(S_{F_x}) \leq 2Aq^n\rho_n$, for any $n \geq 0$. However, since $q > 1$, $q^n\rho_n < \bar{\rho}_n = \sum_{k=n}^{\infty} q^k h_k$. Because of the stability assumption, $\lim_{n\to\infty} \bar{\rho}_n = 0$. Hence $\lim_{n\to\infty} q^n \rho_n = 0$ and $L(S_{F_x}) = 0$. This shows that x_k is a singular RV.

Let us present some consequences of this theorem.

- Consider the case of the IR $h_k = a^k u_k$ giving $\rho_n = a^n/(1-a)$. As a result $L(S_{F_x}) = 0$ as soon as $a < 1/q$. The previous result for SBWN is again obtained for $q = 2$ and $A = 1$. Furthermore, the limit is reached for $a = 1/2$. This is because $\rho_n = 2(2^{-n})$ and $L(S_{F_x}) = 4$.
- In our simulations we shall consider the filter with the IR $h_k = (k+1)a^k u_k$. Simple algebra yields $\rho_n = a^n(\alpha + n\beta)$ where α and β are constants. Therefore $L(S_{F_x}) = 0$ as soon as $a < 1/q$ (the same condition as previously). However, Theorem 1 gives a more restrictive result since $S < 2h_0$ yields $a < 1 - 2^{-1/2} \approx 0.2929$.
- **Circle of singularity**. Consider an ARMA signal x_k generated by a q-valued white noise w_k and a dynamical filter. The stability condition is ensured if all the poles of its transfer function (TF) $H(z)$ are located inside the unit circle, called *circle of stability*. The same discussion can be presented for the filter with IR $g_k = q^k h_k$. Its transfer function is $G(z) = H(z/q)$. One can associate the pole $q\pi_i$ of $G(z)$ to any pole π_i of $H(z)$. The stability condition of Theorem 2 is ensured if $q|\pi_i| < 1$, i.e. if all the poles of $H(z)$ are inside the circle of center 0 and radius $1/q$. This circle is called *circle of singularity* by analogy to the circle of stability.

The location of poles inside the unit circle is a *necessary and sufficient* condition for stability. However, the singularity circle provides only a sufficient condition ensuring singularity. For example, it is possible to obtain singularity for any value of $a \in]1/2, 1[$, for the filter $h_k = a^k u_k$ driven by a non-symmetric BWN by using appropriate probability masses for the RVs w_k. There also exist values of $a \in]1/2, 1[$ yielding singularity for a SBWN. However, the set of such points is of zero measure [8] and can be ignored in a discussion concerning Signal Theory. Note also that Theorem 2 is independent of the values of the IID RVs w_k and of their probabilities. Theorem 2 depends only upon the *number* q of discrete values of the input.

2.2. Fractal properties

Consider the random partial sum $x_{[n]} = \sum_{k=0}^{n-1} h_k w_k$, where w_k is a symmetric SBWN. $x_{[n]}$ takes on 2^n values $v_i^n = \sum_{k=0}^{n-1} h_k \epsilon_k$ where $\epsilon_k = \pm 1$ and $x_{[n]}$ is symmetric. Finally, as in Theorem 1, assume that $h_k > 0$. Indeed, as $\epsilon_k = \pm 1$, the possible values v_i^n are the same if we replace h_k by $|h_k|$.

It is possible to associate two values v_i^{n+1} as defined by $v_i^n \pm h_n$ to each value v_i^n. Repeating this procedure, a tree can be constructed whose nodes are the positive values v_i^n of the RV $x_{[n]}$. These are represented in Figure 1 where only the nodes generated by h_0 are shown. Of course, there is a symmetric tree starting from $-h_0$.

The condition of Theorem 1, $h_n > \rho_{n+1}$, implies that there is no crossing of the branches of the tree. Consider first the branches starting from $\pm h_0$. There is no crossing if the nodes generated by h_0 are all positive. By symmetry, this implies that the nodes generated by $-h_0$ are all negative. This is realized if $h_0 - \rho_1 > 0$, which is the condition of Theorem 1 for $n = 0$. The same procedure can be applied starting from any node v_i^n of the tree. There is no crossing of branches of the tree coming from an arbitrary node v_i^n if $v_i^n - h_n + \rho_{n+1} < v_i^n + h_n - \rho_{n+1}$. This yields the condition $h_n > \rho_{n+1}$. When $h_k = a^k u_k$, $0 < a < 1/2$, this non-crossing property yields an infinite repetition of the same structure because of the properties of the exponential.

The non-crossing property implies the presence of *holes* in the histogram of x_k, i.e. domains where the DF is constant. Indeed, the interval $[v_i^n - h_n + \rho_{n+1}, v_i^n + h_n - \rho_{n+1}]$ with positive measure $2(h_n - \rho_{n+1})$ contains no point of variation of $F_x(\xi)$ and produces a hole in the histogram of x_k when the conditions of Theorem 1 are satisfied.

The non-crossing property can be valid only for sufficiently large n, or if $n > N$. The properties of symmetries and holes are also valid with this condition and are thus asymptotic.

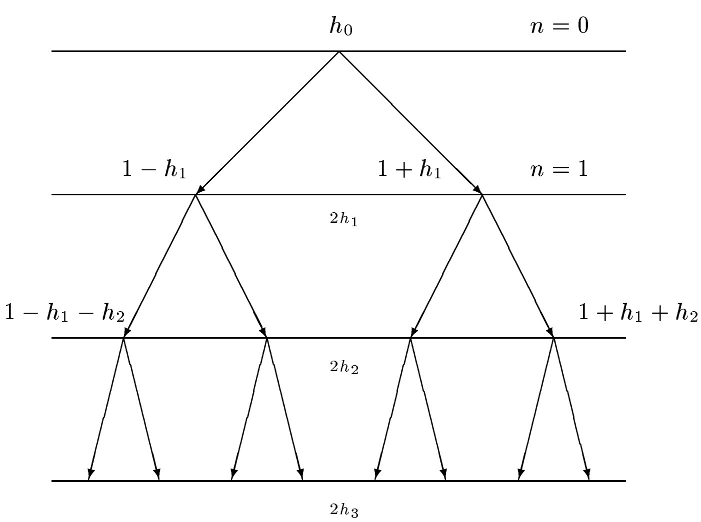

Fig. 1. Tree of successive possible values v_i^n.

3. SIMULATIONS WITH AR(1) SIGNALS

An AR(1) signal is defined by the recursion $x_k = ax_{k-1} + w_k$, $k > 0$. Thus, an AR(1) signal is characterized by the regression coefficient a, the DF $F_w(\xi)$ of the IID RVs w_k, and by the initial value x_0. The influence of x_0 is important only for the small values of k and can be neglected for $k > 10$. Note that the recursion generating x_k corresponds to a filter (2) with $h_k = a^k u_k$. The driving noise w_k is a SBWN in all simulations of this paper.

The appropriate tool for analyzing the DF of the signal x is the histogram. Starting from N samples $x_k, k = 1, ..., N$ the purpose of the histogram is to record the number $n(\alpha, \beta)$ of such samples satisfying the condition $\alpha < x_k \leq \beta$. The difference $\beta - \alpha$ is the length of the analysis cell. For large values of N the ratio $n(\alpha, \beta)/N$ gives an estimate of the increment $F(\beta) - F(\alpha)$ of the DF $F(\xi)$. This result (known for IID RVs) remains valid when the samples are correlated with the correlation function of AR signals. The normalized histogram yields an evaluation of the PDF for small values of the difference $\beta - \alpha$ when the DF has a derivative. However, since the DF can be without a PDF, histograms at different scales have to be realized by using the procedure described below.

The histograms of AR(1) signals obtained from 2.5×10^6 samples x_k are depicted in Figure 2 for different values of a (1: $a = 0.2$; 2: $a = 1/3$; 3: $a = 0.4$; 4: $a = 0.5$). The values of x_k are classified in 400 equal width adjacent cells covering an interval $[-B, +B]$ where $B = 1.1S$ (S is the sum defined by (3)). Histogram 2.1 shows only a small number of apparent symmetries. The symmetries appear more clearly in the other histograms. Finally the last histogram for $a = 1/2$ corresponds to the theory that predicts a uniform distribution in the interval $[-2, +2]$. Note that S is the maximum value of v_i^n. Thus $F(\xi) = 1$ when $\xi \geq S$. The values of S appearing in Fig. 2 are 1.25, 1.5, 1.667, and 2 since for AR signals $S = 1/(1 - a)$.

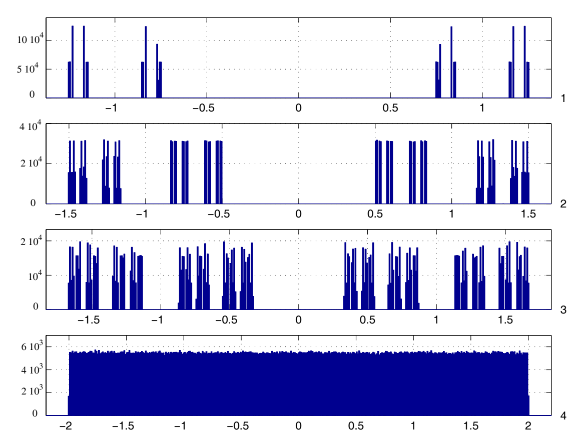

Fig. 2. Histograms of x_k for various a.

Histogram 2 of Fig. 2 (corresponding to $a = 1/3$) is analyzed at different scales in order to better understand the singularity. The center of the histogram is chosen at one of the symmetry centers, i.e. at $c_n = \sum_{k=0}^{n-1} h_k$ (a node of the tree in Fig. 1) in order to highlight the symmetries of the DF. The intervals of analysis of each histogram are defined as $[c_n - 1.1\rho_n, c_n + 1.1\rho_n]$ (ρ_n is defined by (4)) to highlight the fractal autosimilarity. The results are presented in Figure 3.

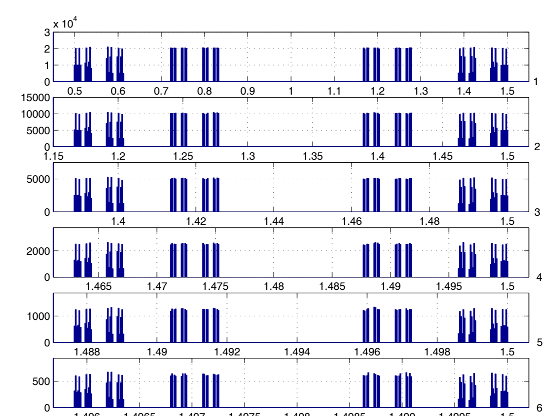

Fig. 3. Histograms of x_k at differents scales for $a = 1/3$.

The symmetry centers (calculated for $a = 1/3$) are 1, 1.333, 1.444, 1.4815, 1.4938, 1.4979. The six histograms of Figure 3 correspond to the analysis of 2.6×10^6 successive samples of the signal x_k. The phenomenon of autosimilarity is especially remarkable. Note that histogram 6 of Figure 3 corresponds to the interval [1.4958, 1.5] of the second histogram of Figure 2 to appreciate the precision of the result. Thus, the structure of the histograms remains almost the same, in spite of a scaling effect of the order of 700. This corresponds to the generation of the nodes v_i^n illustrated by the tree of Figure 1. It also explains the origin of the singularity of the signal. Indeed, this figure can be realized for any node v_i^n of the tree. Continuing with the collection of histograms for $n \to \infty$, any interval $\Delta\xi$ does not contain points of variation of the DF $F(\xi)$, which means that the derivative is almost everywhere equal to zero, i.e. that x_n is singular.

4. AR SIGNALS WITH MULTIPLE POLES

The previously analyzed AR(1) signals are generated from the SBWN w_k by the filter (2) with the transfer function $H_1(z) = z/(z-a)$. To better understand the singularity phenomenon, consider AR signals x_k generated by filters with transfer functions $H_p(z) = [H_1(z)]^p$, introducing a pole of order p in a, driven by the same input white noise w_k.

The histograms of the signals x_k are represented in Figure 4 for $p = 1, 2, 3$ and $a = 0.4$. For this value of a, the conditions of Theorem 1 are satisfied for $p = 1$ (simple pole) but not for $p = 2$ (double pole) and $p = 3$ (triple pole). The sum S defined by (3) can be expressed as $S = 1/(1-a)^p$. Its values for $p = 1$, 2, 3 are 1.6667, 2.7778, 4.6296, which appear clearly in the figure. Note that the central hole in the histogram (analyzed previously in the framework of fractal structures) appears only for $p = 1$. This arises because the conditions of Theorem 1 are not satisfied for $p = 2$ and $p = 3$. Furthermore, the signals x_k seem to have a PDF for double or triple poles. However, this is only an impression. Figure 5 shows histograms of the signal x_k corresponding to a double pole as in Figure 4.2 at different scales. The fractal structure with central hole in the histogram begins to appear with histogram 4. This shows that the singularity cannot be observed only from a global histogram and requires a more detailed analysis. The same phenomenon appears in the case of the triple pole but is not discussed here.

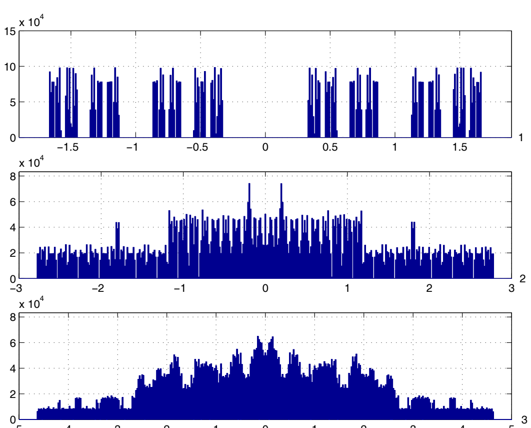

Fig. 4. Histograms of x_k for $n = 1, 2, 3$, $a = 0.4$.

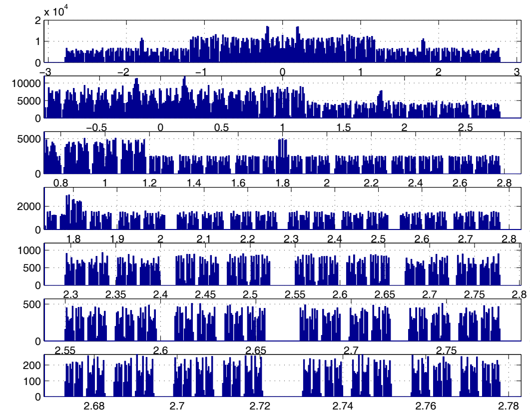

Fig. 5. Histograms for double pole at different scales.

5. CONCLUSION

This paper introduced a new class of signals called *singular random signals*. These signals are characterized by purely singular instantaneous DFs. Singular random signals are usually considered as having only a mathematical interest without impact on Signal Theory. To the contrary, this paper has shown that very simple random signals can exhibit singularity. For example, ARMA signals driven by discrete inputs are singular when the poles are located inside the singularity circle. The singularity circle has the same center as the stability circle but with radius $1/q$ (q is the number of discrete values of the input). This radius is 1/2 for Bernoulli white noise with two possible values. The fractal properties of the DF were discussed and illustrated by histograms at different scales obtained with AR(1) and multiple pole AR signals.

This work can be extended in various ways to better understand the properties of singular signals. Similar results were obtained with simulations of more general ARMA signals (including signals with complex poles). The condition for the pole locations is only sufficient. Thus, it is of great interest to extend the domain of singularity. Finally, it is of great interest to analyze the effects of the singularity on various Statistical Signal Processing procedures, especially those where the probability density function is used.

6. REFERENCES

[1] M. Loève, *Probability Theory*. New-York: Springer Verlag, 1977.

[2] E. Wong, B. Hajek, *Stochastic Processes in Engineering Systems*. New-York: Springer Verlag, 1985.

[3] E. Lukacs, *Characteristic Functions*. London: Griffin, 1970.

[4] A. Papoulis, *Probability, Random Variables and Stochastic Processes*. New-York: Mc Graw Hill, 1984.

[5] C.W. Helstrom, *Probability and Stochastic Processes for Engineers*, New-York: Macmillan, 1991.

[6] B. Picinbono, *Random Signals and Systems*, Englewood Cliffs, NJ: Prentice Hall, 1993.

[7] J.-Y. Tourneret and B. Lacaze, "Étude et simulation des lois de probabilité de sorties de modèles paramétriques," *Traitement du Signal*, vol. 11, pp. 271-281, 1994.

[8] Y. Peres, W. Schlag and B. Solomyak, "*Sixty years of Bernoulli convolutions*," Fractals and Stochastics II, (C. Bandt, S. Graf and M. Zaehle, eds.), Progress in Probability, vol. 46, pp. 39-65, Birkhauser, 2000.