CONSISTENCY ANALYSIS OF A FREQUENCY DOMAIN SUBSPACE ALGORITHM FOR MULTI-COMPONENT HARMONIC RETRIEVAL

Joakim Gunnarsson and Tomas McKelvey

Department of Signals and Systems, Chalmers University of Technology
SE-412 96 Göteborg, Sweden, Ph:+46 (0)31 772 2140
joakim.gunnarsson@s2.chalmers.se, tomas.mckelvey@s2.chalmers.se

ABSTRACT

A new frequency domain algorithm for multi-component harmonic retrieval has been developed recently. The method is based on using Discrete Fourier Transform (DFT) data and enables the user to select frequency sub-bands within which the frequencies of the harmonics of interest are known to reside. The technique provides a very good attenuation of colored noise and other disturbances residing outside the frequency band of interest. This paper presents a consistency analysis of the subspace based estimation algorithm. The analysis provides sufficient conditions for consistency of the sinusoidal frequency estimates in terms of required number of DFT data. A small numerical example is also presented comparing the new approach with the well known ESPRIT method. The result clearly indicates that the method has a large potential.

1. INTRODUCTION

The number of applications for estimating frequencies of sinusoids is great and as a result, the research field is old, popular and to a certain degree mature. A large number of algorithms have been developed for this problem. Due to various requirements such as frequency resolution, computational complexity and data quality, the preferable algorithm may vary. A special family of frequency estimation methods which has been studied thoroughly are subspace based methods. These have been proven to yield consistent estimates under certain conditions [1, 2, 3, 4]. In this paper we are studying consistency properties for a recently proposed algorithm [5]. This algorithm is here referred to as F-ESPRIT, due to its similarities to the ESPRIT algorithm derived in [3] and since it uses frequency domain data.

The classical problem we are studying is the estimation of n harmonic components in signal models of the type

$$\mathbf{y}(t) = \sum_{k=1}^{n} \boldsymbol{\alpha}_k e^{(\sigma_k + i\omega_k)t}, \tag{1}$$

where $\mathbf{y}(t) \in \mathbb{C}^m$ is a measured signal for $t = 0, .., N - 1$. Other than the signal frequencies, $\omega_k \in \mathbb{R}$, estimations of damping factors, $\sigma_k \in \mathbb{R}$, and gains, $\boldsymbol{\alpha_k} \in \mathbb{C}^m$, are also sought for. The analysis will be based on the following assumption on the signal parameters:

Assumption 1
The complex frequencies $\lambda_k \triangleq \sigma_k + i\omega_k$ *are all distinct and* $\forall k$, $\boldsymbol{\alpha_k} \neq \mathbf{0}$ *and* $\omega_k \in (-\pi, \pi)$.

This paper is divided into four sections. Section 2 shortly reviews the derivation of the F-ESPRIT estimation method and introduces the notation. After that follows the main contribution which is conditions for consistency of the frequency estimates. In the last section a numerical example is presented.

2. F-ESPRIT

The idea of F-ESPRIT is to incorporate a-priori information about frequency intervals in which the unknown frequencies are expected to reside. Using the DFT the sampled data is converted to the frequency domain. Only DFT data in specified frequency intervals are used in the estimation. Disturbances outside the frequency band are then naturally suppressed, since only leakage into the interval affects the estimates. The method has successfully been used for analysis of electromagnetic time domain simulation data [6, 7].

First the time domain model is written in state space form.

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \quad \mathbf{x}(0) = \mathbf{x_0} \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t), \end{aligned} \tag{2}$$

with the following matrix notation

$$\mathbf{A} = \mathrm{diag}[e^{\lambda_1}, e^{\lambda_2}, \cdots, e^{\lambda_n}] \in \mathbb{C}^{n \times n} \tag{3}$$

$$\mathbf{C} = [\boldsymbol{\alpha}_1 \quad \boldsymbol{\alpha}_2 \quad \cdots \quad \boldsymbol{\alpha}_n] \in \mathbb{C}^{m \times n} \tag{4}$$

$$\mathbf{x_0} = [1 \quad 1 \quad \cdots 1]^T \in \mathbb{R}^n. \tag{5}$$

The extra signal $u(t)$ and a particular choice of the $\boldsymbol{B}$ matrix defined by

$$u(t) \triangleq \begin{cases} 1, & t = kN-1, \quad k = 1,2,\cdots \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

$$\mathbf{B} \triangleq (\mathbf{I} - \mathbf{A}^N)\mathbf{x_0}. \tag{7}$$

makes the state vector $\mathbf{x}(t)$ an N-periodic sequence, i.e., $\mathbf{x}(t+N) = \mathbf{x}(t)$. It should be noted that in the observed interval $t = 0, \cdots, N-1$, the two model descriptions (1) and (2) are identical. A frequency domain equivalent model is retrieved with the DFT, which is defined by

$$\mathbf{x}_k \triangleq DFT\{\mathbf{x}(t)\}_k \triangleq \sum_{t=0}^{N-1} \mathbf{x}(t) W_N^{kt} \triangleq \sum_{t=0}^{N-1} \mathbf{x}(t) e^{j\frac{2\pi k}{N}t}. \tag{8}$$

A known property of the Fourier Transform is that a time shift corresponds to a phase shift in the frequency domain. For the DFT, that only holds if the signal is periodic and the measured signal contains an exact multiple of the period. The periodicity of $\mathbf{x}(t)$ now results in $DFT\{\mathbf{x}(t+1)\}_k = W_N^k \mathbf{x}_k$ and a frequency domain equivalent of model (2) takes the form

$$\begin{aligned} W_N^k \mathbf{x}_k &= \mathbf{A}\mathbf{x}_k + \mathbf{B}u_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k, \end{aligned} \tag{9}$$

where $\mathbf{y}_k \triangleq DFT\{\mathbf{y}(t)\}_k$ and $u_k \triangleq DFT\{u(t)\}_k = W_N^k$. From here, an expression is sought which enables subspace based estimation. We use equation (9) to, for DFT frequency k, form the vector relation

$$\mathbf{Y}_k = \mathcal{O}_s \mathbf{x}_k + \mathbf{\Gamma_s} \mathbf{u}_k \tag{10}$$

where,

$$\mathbf{Y}_k \triangleq \begin{bmatrix} \mathbf{y}_k \\ W_N^k \mathbf{y}_k \\ W_N^{2k} \mathbf{y}_k \\ \vdots \\ W_N^{(s-1)k} \mathbf{y}_k \end{bmatrix}, \quad \mathbf{u}_k \triangleq \begin{bmatrix} W_N^k \\ W_N^{2k} \\ W_N^{3k} \\ \vdots \\ W_N^{sk} \end{bmatrix}, \tag{11}$$

$$\mathcal{O}_s \triangleq \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{s-1} \end{bmatrix} \tag{12}$$

and

$$\mathbf{\Gamma_s} \triangleq \begin{bmatrix} \mathbf{0} & & & & \\ \mathbf{CB} & \mathbf{0} & & & \\ \mathbf{CAB} & \ddots & \ddots & \ddots & \\ \vdots & & & \ddots & \\ \mathbf{CA}^{s-2}\mathbf{B} & \mathbf{CA}^{s-3}\mathbf{B} & \cdots & \mathbf{CB} & \mathbf{0} \end{bmatrix}. \tag{13}$$

The parameter s is an auxiliary parameter in subspace methods and decides the number of rows used in the matrices in equation (10). Its influence on the estimates is not discussed in this article, it must however be chosen larger than the number of harmonics, n.

Now a size M subset of the frequencies on the DFT grid is picked out. These frequency indices are denoted by k_i and are used to form the data matrices

$$\mathbf{Y} \triangleq [\mathbf{Y}_{k_1} \quad \mathbf{Y}_{k_2} \quad \cdots \quad \mathbf{Y}_{k_M}] \tag{14}$$

$$\mathbf{U} \triangleq [\mathbf{u}_{k_1} \quad \mathbf{u}_{k_2} \quad \cdots \quad \mathbf{u}_{k_M}] \tag{15}$$

$$\mathbf{X} \triangleq [\mathbf{x}_{k_1} \quad \mathbf{x}_{k_2} \quad \cdots \quad \mathbf{x}_{k_M}]. \tag{16}$$

The signal model is now written as a matrix relation

$$\mathbf{Y} = \mathcal{O}_s \mathbf{X} + \mathbf{\Gamma_s} \mathbf{U}, \tag{17}$$

which forms the basic subspace equation with a structure common for many subspace identification methods [1]. The first term on the right side contains all the desirable information and by construction has a rank equal to n, the number of signal components. The second term is removed by a multiplication from the right by a projection matrix defined as

$$\mathbf{\Pi}^{\perp} = \mathbf{I} - \mathbf{U}^*(\mathbf{U}\mathbf{U}^*)^{-1}\mathbf{U}. \tag{18}$$

The resulting equation from which the frequencies and amplitudes can be retrieved, with e.g. Kung's algorithm [4], is

$$\mathbf{Y}\mathbf{\Pi}^{\perp} = \mathcal{O}_s \mathbf{X}\mathbf{\Pi}^{\perp}. \tag{19}$$

The estimation of the frequencies will be consistent if the resulting right hand term does not lose any rank by the multiplication of the projection matrix [8, 4, 3, 1]. This will be the topic of the next section.

3. ANALYSIS

The following section contains the main contribution of the paper. The exact conditions which ensure that the F-ESPRIT method consistently can identify the signal parameters are given in the result below.

Theorem 1
Let Assumption 1 hold and let $\{k_i, i = 0, ..., M-1\}$ be a subset of the index set $\{0, ..., N-1\}$, where M is the number of frequency domain data and where $\mathbf{y}_{k_i} \neq \mathbf{0}$ for at least n of the indices k_i. When the parameter s is chosen such that $M \geq n + s$ and $s > n$, the estimates of the parameters $\boldsymbol{\alpha}_k$, σ_k and ω_k given by F-ESPRIT [5] will be consistent.

Proof: The only part of the algorithm that needs to be examined is the influence of the projection matrix $\mathbf{\Pi}^{\perp}$ on the rank of $\mathcal{O}_s \mathbf{X}\mathbf{\Pi}^{\perp}$. First, Assumption 1 together with $\mathbf{y}_{k_i} \neq 0$ for at least n indices imply that $\mathbf{X}$ has full rank n. To retrieve the parameters, the projection $\mathbf{\Pi}^{\perp}$ must not cancel any part of the range space of $\mathbf{X}$. This cancellation would only occur if and only if the rows of $\mathbf{U}$ and $\mathbf{X}$ are linearly dependent or if the matrix $\mathbf{\Upsilon} = \begin{bmatrix} \mathbf{U} \\ \mathbf{X} \end{bmatrix}$ is rank deficient. The matrix $\mathbf{\Upsilon}$ is rank deficient if and only if there exists a vector

$$[a_1 \quad \cdots \quad a_s \quad b_1 \quad \cdots \quad b_n] = \begin{bmatrix} \mathbf{a}^T & \mathbf{b}^T \end{bmatrix} \neq \mathbf{0}, \quad (20)$$

such that

$$\begin{bmatrix} \mathbf{a}^T & \mathbf{b}^T \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{X} \end{bmatrix} = 0. \quad (21)$$

Let the $\mathbf{A}$ and $\mathbf{B}$ matrices in (9) be arranged into two disjoint sets corresponding to the p complex frequencies on the DFT grid, i.e., $e^{\lambda_q} = W_N^{l_q}, \quad q = 1, \cdots, p$, and the ones off the grid. Note that the case $p > 0$ corresponds to a thin subset of the signal model set.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_1} & 0 \\ 0 & \mathbf{A_2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B_1} \\ \mathbf{B_2} \end{bmatrix} \quad (22)$$

Here $\mathbf{A_1}$ and $\mathbf{B_1}$ corresponds to frequencies off the DFT-grid and $\mathbf{A_2}$ and $\mathbf{B_2}$ on the grid. Note that $\mathbf{B_2} = \mathbf{0}$ since $\mathbf{A_2}^N = \mathbf{I}$. The matrix $\mathbf{X}(z)$ now takes the form

$$\begin{aligned} \mathbf{X}(z) &= \begin{bmatrix} \mathbf{X_1}(z) \\ \mathbf{X_2}(z) \end{bmatrix} \\ &= \begin{bmatrix} (z\mathbf{I} - \mathbf{A_1})^{-1}\mathbf{B_1} z \\ \mathbf{X_2}(z) \end{bmatrix}. \end{aligned} \quad (23)$$

Here the matrix $\mathbf{X_2}(z)$ is only nonzero for $z = W_N^{l_q}$ and can be written as

$$\mathbf{X_2}(z) = N \begin{bmatrix} \delta(z - W_N^{l_1}) \\ \vdots \\ \delta(z - W_N^{l_p}) \end{bmatrix} \quad (24)$$

The function $\delta(x)$ is Kronecker's δ-function which is defined as

$$\delta(x) \triangleq \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}.$$

Now introduce

$$\mathbf{U}(z) = \begin{bmatrix} z & z^2 & z^3 & \ldots & z^s \end{bmatrix}^T \quad (25)$$

and

$$H(z) = \mathbf{a}^T\mathbf{U}(z) + \tilde{\mathbf{b_1}}^T\mathbf{X_1}(z) + \tilde{\mathbf{b_2}}^T\mathbf{X_2}(z). \quad (26)$$

Hence, equation (21) can be formulated as

$$H(z) = 0, \quad z = W_N^{k_1}, ..., W_N^{k_M}. \quad (27)$$

First study those values of the discrete variable z for which $z \neq W_N^{l_q}, q \in [1, \cdots, p]$. Equation (26) then takes the form

$$\begin{aligned} \tilde{H}(z) = \mathbf{a}^T\mathbf{U}(z) + \tilde{\mathbf{b_1}}^T\mathbf{X_1}(z) = 0, \\ z = W_N^{l_q}, \quad q = p+1, \cdots, M \end{aligned} \quad (28)$$

Notice that $\tilde{H}(z)$ is a rational function of z and to fulfill the zero product, it must contain $M-p$ zeros. The degree of the numerator polynomial in $\tilde{H}(z)$ is however only $n-p+s-1$ and since M and s satisfy $M \geq n+s$, $\tilde{H}(z)$ does not have enough zeros. Therefore equation (21) is only fulfilled if

$$\tilde{H}(z) = \mathbf{a}^T\mathbf{U}(z) + \tilde{\mathbf{b_1}}^T(z\mathbf{I} - \mathbf{A_1})^{-1}\mathbf{B_1} z \equiv 0 \quad (29)$$

The terms in (29) can be separated since they correspond to different polynomial coefficients; the first term only involves the s highest powers and so forth.

$$\begin{aligned} \mathbf{a}^T\mathbf{U}(z) &\equiv 0, \quad \Rightarrow \quad \mathbf{a} = \mathbf{0} \\ \tilde{\mathbf{b_1}}^T(z\mathbf{I} - \mathbf{A_1})^{-1}\mathbf{B_1} &\equiv 0 \end{aligned} \quad (30)$$

From system theory [9] it is known that equation (30) only holds for a nonzero $\tilde{\mathbf{b_1}}$ if and only if $(\mathbf{A_1}, \mathbf{B_1})$ is a non-controllable pair, or the controllability matrix $\boldsymbol{\mathcal{C}}$ has full rank.

$$\boldsymbol{\mathcal{C}} = [\mathbf{B_1} \quad \mathbf{A_1}\mathbf{B_1} \quad \cdots \quad \mathbf{A_1}^{n-p-1}\mathbf{B_1}] \quad (31)$$

$$= \begin{bmatrix} 1 - e^{N\lambda_{p+1}} & e^{\lambda_{p+1}}(1 - e^{N\lambda_{p+1}}) & \cdots \\ \vdots & \vdots & \\ 1 - e^{N\lambda_n} & e^{\lambda_n}(1 - e^{N\lambda_n}) & \cdots \end{bmatrix} \quad (32)$$

The matrix $\boldsymbol{\mathcal{C}}$ can be factorized into a diagonal and a Vandermonde matrix and it is easy to establish, see e.g. [9] that under Assumption 1 both these matrices are of full rank. It is therefore concluded that equation (29) is only true if and only if

$$\begin{aligned} \mathbf{a} &= \mathbf{0} \\ \tilde{\mathbf{b_1}} &= \mathbf{0}. \end{aligned} \quad (33)$$

The remaining part of equation (26) is now

$$\tilde{\mathbf{b_2}}^T\mathbf{X_2}(z) = 0, \quad (34)$$

which should hold for all $z = W_N^k$. Each component of $\mathbf{X_2}(z)$ is non zero for one of the p remaining frequencies. Which directly implies $\tilde{\mathbf{b_2}} = \mathbf{0}$. For the rest of the proof see e.g. [8].

□

The result shows that the F-ESPRIT method consistently can estimate all harmonic components subject to that the number of data is larger or equal to $N = 2n + 1$.

4. EXAMPLE

In this example, the algorithm was tested on a signal with a frequency spectra containing a large number of harmonics. The signal is from a numerical solution of Maxwell's equations when an object is subject to a broadband excitation. It is a real valued signal consisting of 1371 sinusoids. In Figure 1 the periodogram with a Chebyshev window shows the spectral content of this signal.

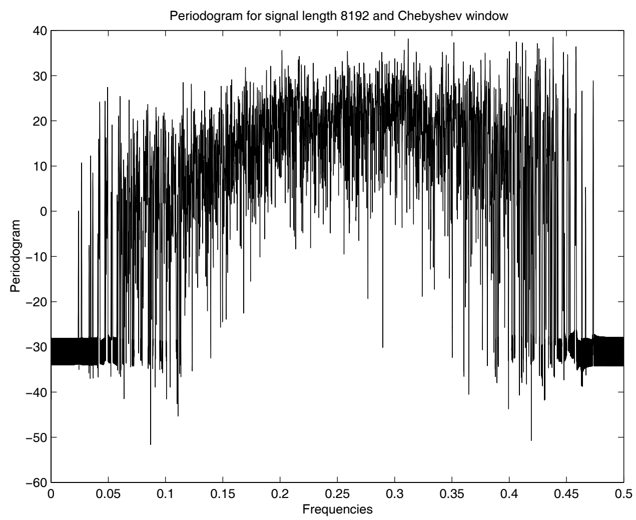

Fig. 1. Periodogram of the simulated signal calculated from 8192 time domain samples using a Chebyshev window.

From this signal, frequency estimates of the five lowest harmonics are sought for. F-ESPRIT is used with data in the normalized frequency interval $\{0.019, 0.038\}$. Using only N=1024 samples the 5 frequencies are estimated with a maximum absolute error of 4.3×10^{-5}. To get an idea of the quality of the estimates we compare them with time domain ESPRIT for varying sizes of the data length N. To successfully use ESPRIT in this scenario filtering with a sharp low-pass filter is necessary to suppress the harmonics above the five first. Hence, the signal is low-pass filtered with a Butterworth filter of order 17 with a cut-off frequency of 0.04. The estimates are then compared by studying the ratio between the absolute error for each method. These results are shown in Figure 2, where each subplot contains the ratio for the different frequencies. From the figures it is seen that F-ESPRIT outperforms ESPRIT in this scenario.

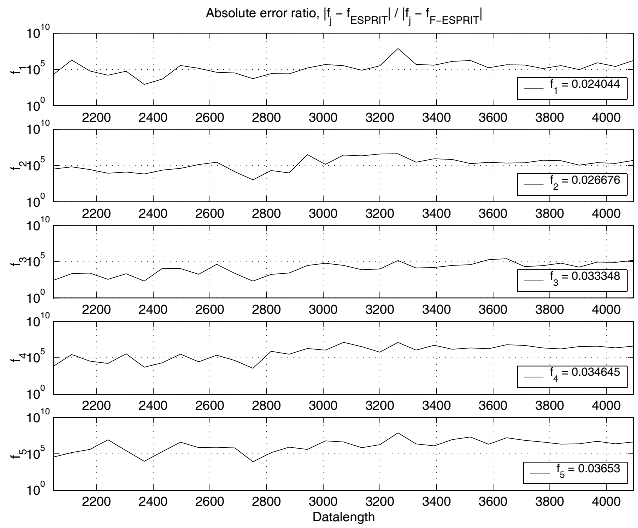

Fig. 2. Absolute error ratio between the ESPRIT and F-ESPRIT methods.

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