DISCRETE-TIME ANALYTIC SIGNALS WITH IMPROVED SHIFTABILITY

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ABSTRACT

In this paper we consider a common procedure [2] for generating analytic signals and show how it fails for specific discrete-time real signals. A new frequency domain technique is formulated that solves the defect. Both methods have the same redundancy. The new analytic signal preserves the original signal (real part) as also the zeros of its discrete spectrum in the negative frequencies. The superiority of the new method is in the introduction of one additional zero of the continuous spectrum of the original signal at a negative frequency and a corresponding reduction in shiftability.

1. INTRODUCTION

A continuous-time analytic signal is a complex time function having a Fourier transform equal to zero for all $\omega < 0$. For a complex time sequence, we cannot require the same constraint since the discrete Fourier transform (DFT) spectrum is periodic. Instead, a complex sequence is defined to be "analytic" [3] by requiring its discrete-time Fourier transform (DTFT) vanish in the interval $[-\pi, 0)$. Such a sequence will henceforth be denoted as a discrete-time analytic (DTA) signal. The motivation for transforming a real valued signal to an analytic one stems from the fact that the negative frequencies are conjugate symmetric of the positive ones. Thus information contained therein is redundant. Hence removing negative frequencies while preserving positive components, will not affect the information contained in the new signal. The advantage of processing DTA signals is seen in many applications: for example, in discrete wavelet transforms, we see reduction of shift sensitivity and an improved directionality [1], Spectral analysis[7], [8] we see estimation of instantaneous frequency. Methods currently used to generate DTA signals are either time domain [4],[6] filtering methods or frequency-based ones [2]. The former consists of a lowpass filter design and a spectrum shift to the right, maintaining optimal attenuation in the negative frequency band. The frequency domain approach consists of setting the negative frequencies to zero via the discrete Fourier transform (DFT), and subsequent generation of the DTA through an inverse DFT. In the time domain approach, the length of the filter affects the accuracy of the approximation to the analytic signal. For instance in [4], the number of taps used is 128 which make this method inappropriate for small-length signals. We show in Section 2 that for a specific class of signals, the frequency domain algorithm fails by generating signals with zero imaginary part.

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In this paper, we review the method used in [2] for DTA function generation. An alternative method based on the DTFT spectrum is developed. The two techniques are compared for the degree of aliasing generated by measuring their shiftability [5].

2. DISCRETE ANALYTIC SIGNAL VIA DFT

Oppenheim, et al. [3] formulated the concept of a DTA signal: For $x(n)$ a finite real valued sequence, the DTA signal $z(n)$ is defined as,

$$z(n) = x(n) + jH\{x(n)\}$$

where H is the Hilbert transform operator, and $j^2 = -1$. The periodic spectrum of $z(n)$ is,

$$Z(e^{j\omega}) = \sum_{n=0}^{N-1} z(n)e^{-j\omega n}$$

where N is the length of $z(n)$. The DTFT is periodic with period 2π. Thus ω is considered in the interval$[-\pi, \pi]$. For analyticity it is required [[3], Sec.10.4] that $Z(e^{j\omega}) = 0$ for $\omega \in [-\pi, 0)$. Because ω is continuous in the interval $[-\pi, \pi]$, the DTFT cannot be computed exactly. Hence the necessity for using the DFT. The DFT is obtained by uniformly sampling the DTFT on the ω-axis at $\omega_k = 2\pi k/N$, where $0 \leq k \leq N-1$. Thus,

$$Z(k) = Z(e^{j\omega})|_{\omega=\frac{2\pi k}{N}} = \sum_{n=0}^{N-1} z(n)e^{-j2\pi kn/N}$$

A common approach to generating a DTA signal resides in the frequency domain [2]. We assume N to be even (the case for N odd is easily handled). The procedure for deriving the discrete analytic signal consists of three steps:

- Compute the N-point DFT of $x(n)$.
- Form the N-point DFT of the discrete analytic signal by multiplying the N-point DFT of $x(n)$ by the vector:

$$a(n) = \begin{cases} 1, & n = 0. \\ 2, & 1 \leq n \leq N/2 - 1. \\ 1, & n = N/2. \\ 0, & N/2 + 1 \leq n \leq N - 1 \end{cases}$$

Thus, the N-point DFT of the discrete analytic signal is,

$$Z(k) = \begin{cases} X(k), & k = 0. \\ 2X(k), & 1 \leq k \leq N/2 - 1. \\ X(k), & k = N/2. \\ 0, & N/2 + 1 \leq k \leq N - 1 \end{cases}$$

- Obtain the discrete analytic signal by computing the inverse DFT of the N-point DFT :

$$z(n) = 1/N \sum_{k=0}^{N-1} Z(k) e^{j2\pi kn/N}$$

Based on this procedure, the analytic function for n even, is seen to be,

$$\begin{aligned} z(n) &= x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p+1) \\ & \quad cot(\pi(n-(2p+1))/N) \end{aligned} \tag{1}$$

and for n odd,

$$z(n) = x(n) + j(2/N) \sum_{p=0}^{N/2-1} x(2p) cot(\pi(n-2p)/N) \tag{2}$$

As stated earlier, a continuous analytic signal is a complex function; thus a non-zero real signal is not analytic because of the relationship between the positive and negative frequency. Similarly a non-zero discrete real signal is not analytic. From equations (1) and (2), we deduce that for specific discrete-time real signals, the imaginary part of $z(n)$ is zero; hence the algorithm generates a real signal that is not analytic. For instance, suppose all even values of $x(n)$ are equal to some constant α and all odd values to some constant β (with no loss of generality we assume both constants to be different from zero). Then, for n even, we have,

$$imag(z(n)) = (2/N)\beta \sum_{p=0}^{N/2-1} cot(\pi(n-(2p+1))/N)$$

and for n odd,

$$imag(z(n)) = (2/N)\alpha \sum_{p=0}^{N/2-1} cot(\pi(n-2p)/N).$$

For the imaginary part of $z(n)$ to be zero, we must have,

$$\begin{cases} \sum_{p=0}^{N/2-1} cot(\pi(n-(2p+1))/N) = 0. \\ \sum_{p=0}^{N/2-1} cot(\pi(n-2p)/N) = 0 \end{cases} \tag{3}$$

The conditions in equation (3) are always true (write each equation as a circulant matrix, and show that the sum of the first row is equal to zero). Therefore the procedure results in a signal equal to the original real signal. Hence a DTA is not generated. As an example using MATLAB 6.5, we see that:

$$hilbert([1 \quad 2 \quad 1 \quad 2]) = [1 \quad 2 \quad 1 \quad 2];$$

3. THE NEW METHOD

We have seen an example where the algorithm in [2] fails to generate the corresponding DTA signal. The new method, for which the algorithm in [2] is a special case, is developed in the frequency domain. The procedure entails adding an imaginary number to the DC and the Nyquist terms of the DTA signal obtained using [2]. This guarantees not only that the DTFT equals zero at $\omega_k = 2\pi k/N$, $N/2+1 \leq k \leq N-1$, but also at another negative frequency of our choice. In addition, the real part of the corresponding analytic signal equals the original signal.

Let $x(n)$ be a finite real valued sequence of length N. We restrict our attention to the case where N is even. With reference to equations (1) and (2) let $s(n)$ be a DTA such that, for n even,

$$\begin{aligned} s(n) &= x(n) + j(2/N)\{ \sum_{p=0}^{N/2-1} x(2p+1) \\ & \quad cot(\pi(n-(2p+1))/N) + a\} \end{aligned} \tag{4}$$

and for n odd,

$$s(n) = x(n) + j(2/N)\{ \sum_{p=0}^{N/2-1} x(2p) cot(\pi(n-2p)/N) + b\} \tag{5}$$

where a and b are two real variables. We show that the DFT $S(k)$ of $s(n)$ is zero for $N/2+1 \leq k \leq N-1$. Denote by $z(n)$ the DTA signal given by the algorithm in [2]. Thus, for n even,

$$s(n) = z(n) + j(2/N)a = z(n) + jt(n)$$

and for n odd,

$$s(n) = z(n) + j(2/N)b = z(n) + jt(n)$$

where,

$$t(n) = \begin{cases} a/N & n \ even \\ b/N & n \ odd. \end{cases}$$

The DFT of $t(n)$ is equal to,

$$\begin{aligned} T(k) &= \sum_{n=0}^{N-1} t(n) e^{-j2\pi kn/N} \\ &= \sum_{n=0}^{N/2-1} t(2n) e^{-j2\pi k2n/N} + \\ & \quad \sum_{n=0}^{N/2-1} t(2n+1) e^{-j2\pi k(2n+1)/N} \\ &= 2/N\{a \sum_{n=0}^{N/2-1} e^{-j2\pi k2n/N} + \\ & \quad b \sum_{n=0}^{N/2-1} e^{-j2\pi k(2n+1)/N}\} \end{aligned}$$

Thus,

$$\begin{aligned} S(k) &= Z(k) + jT(k) \\ &= Z(K) + j\{2/N\{a \sum_{n=0}^{N/2-1} e^{-j2\pi k2n/N} + \\ & \quad b \sum_{n=0}^{N/2-1} e^{-j2\pi k(2n+1)/N}\}\} \end{aligned}$$

where $Z(k)$ is the DFT of $z(n)$. We now show that,

$$S(k) = \begin{cases} X(k) + j(a+b), & k = 0 \\ 2X(k), & 1 \leq k \leq N/2-1 \\ X(k) + j(a-b) & k = N/2 \\ 0, & N/2+1 \leq k \leq N-1. \end{cases} \tag{6}$$

Note that $Z(k)$ satisfies the following,

$$Z(k) = \begin{cases} X(k), & k = 0 \\ 2X(k), & 1 \le k \le N/2 - 1 \\ X(k), & k = N/2 \\ 0, & N/2 + 1 \le k \le N - 1. \end{cases}$$

Therefore it is sufficient to show that,

$$T(k) = \begin{cases} a + b, & k = 0 \\ 0, & 1 \le k \le N/2 - 1 \\ a - b, & k = N/2 \\ 0, & N/2 + 1 \le k \le N - 1. \end{cases} \quad (7)$$

The proofs for $k = 0$ and $k = N/2$ are straightforward. For $1 \le k \le N/2 - 1$ we can verify that

$\sum_{n=0}^{N/2-1} e^{-j2\pi k2n/N} = \sum_{n=0}^{N/2-1} e^{-j2\pi k(2n+1)/N} = 0.$

For the remaining interval the proof follows by the conjugate symmetry relationship for real signal. Having established equation (6), values for a and b need to be determined so that the DTFT of $s(n)$ is zero for some ω in the interval $[-\pi, 0)$. Zeroing the continuous DTFT for some ω in the negative frequency range will form a neighborhood of ω where the DTFT could be very small. Hence for such ω we need that,

$$S(e^{j\omega}) = \sum_{n=0}^{N-1} s(n)e^{-j\omega n} = 0. \quad (8)$$

To determine ω we proceed as follows: Let,

$r_1(\omega) = 2/N \sum_{p=0}^{N/2-1} sin(2p\omega).$
$\alpha_1(\omega) = \sum_{p=0}^{N/2-1} x(2p)cos(2p\omega).$
$\alpha_2(\omega) = 2/N \sum_{p=0}^{N/2-1} \sum_{q=0}^{N/2-1} x(2q+1)cot(f(p,q))sin(2p\omega).$
$r_2(\omega) = 2/N \sum_{p=1}^{N/2} sin(\omega(2p-1)).$
$\alpha_3(\omega) = \sum_{p=1}^{N/2} x(2p-1)cos(\omega(2p-1)).$
$\alpha_4(\omega) = 2/N \sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} x(2q)cot(f(p,q))sin(\omega(2p-1)).$
$r_{i1}(\omega) = 2/N \sum_{p=0}^{N/2-1} cos(2p\omega).$
$\alpha_{i1}(\omega) = -\sum_{p=0}^{N/2-1} x(2p)sin(2p\omega).$
$\alpha_{i2}(\omega) = -2/N \sum_{p=0}^{N/2-1} \sum_{q=0}^{N/2-1} x(2q+1)cot(f(p,q))cos(2p\omega).$
$r_{i2}(\omega) = 2/N \sum_{p=1}^{N/2} cos(\omega(2p-1)).$
$\alpha_{i3}(\omega) = -\sum_{p=1}^{N/2} x(2p-1)sin(\omega(2p-1)).$

$$\alpha_{i4}(\omega) = -2/N \sum_{p=1}^{N/2} \sum_{q=0}^{N/2-1} x(2q)cot(f(p,q))cos(\omega(2p-1))$$

where $f(p,q) = (\pi/N)(2p - (2q+1))$. Hence equation (8) implies that,

$$\begin{cases} ar_1(\omega) - br_2(\omega) = \alpha_3(\omega) + \alpha_4(\omega) - \alpha_1(\omega) - \alpha_2(\omega) \\ ar_{i1}(\omega) - br_{i2}(\omega) = \alpha_{i3}(\omega) + \alpha_{i4}(\omega) - \alpha_{i1}(\omega) - \alpha_{i2}(\omega). \end{cases}$$

Accordingly, for $\triangle(\omega) = r_2(\omega)r_{i1}(\omega) - r_1(\omega)r_{i2}(\omega) \ne 0$, we must have,

$$\begin{aligned} a &= \{-r_{i2}(\omega)(\alpha_3(\omega) + \alpha_4(\omega) - \alpha_1(\omega) - \alpha_2(\omega)) + \\ &\quad r_2(\omega)(\alpha_{i3}(\omega) + \alpha_{i4}(\omega) - \alpha_1(\omega) - \alpha_2(\omega))\}/\triangle(\omega) \\ b &= \{r_1(\omega)(\alpha_{i3}(\omega) + \alpha_{i4}(\omega) - \alpha_{i1}(\omega) - \alpha_{i2}(\omega)) - \\ &\quad r_{i1}(\omega)(\alpha_3(\omega) + \alpha_4(\omega) - \alpha_1(\omega) - \alpha_2(\omega))\}/\triangle(\omega) \end{aligned} \quad (9)$$

Now we have,

$$\begin{aligned} \triangle(\omega) &= (2/N)^2\{ \sum_{p=1}^{N/2} sin(\omega(2p-1)) \sum_{q=0}^{N/2-1} cos(2\omega q) - \\ &\quad \sum_{p=1}^{N/2} cos(\omega(2p-1)) \sum_{q=0}^{N/2-1} sin(2\omega q)\} \\ &= (2/N)^2\{\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1} sin(\omega(2p-2q-1))\} \\ &= (2/N)^2\{\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1} imag((e^{j(\omega(2p-2q-1))})\} \\ &= (2/N)^2\{imag(\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1} e^{j(\omega(2p-2q-1)})\}. \end{aligned}$$

We can show that for $\omega \in (-\pi, 0)$,

$$\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1} e^{j(\omega(2p-2q-1)} = j((1 - cos(\omega N)/2sin(\omega)))$$

and for $\omega = -\pi$,

$\sum_{p=1}^{N/2}\sum_{q=0}^{N/2-1} e^{j(\omega(2p-2q-1)} = N/2(N/2-1).$

Therefore, for $\omega \in (-\pi, 0)$,

$\triangle(\omega) = (2/N)^2(1 - cos(\omega N)/(2sin(\omega)))$

and for $\omega = -\pi$,

$\triangle(\omega) = 0$

Thus, for $\omega \in (-\pi, 0)$ we have,

$$\triangle(\omega) = 0 \leftrightarrow cos(\omega N) = 1.$$

Therefore,

$$\triangle(\omega) = 0 \quad \leftrightarrow \quad \omega = 2\pi k/N \text{ for } N/2 + 1 \le k \le N - 1.$$

Letting,

$$A = \begin{cases} \omega \in [-\pi, 0) | \ \omega = 2\pi k/N \\ \text{where } N/2 + 1 \le k \le N - 1 \end{cases}$$

we conclude that for $\omega \in \{(-\pi, 0)/A\}$ the system (9) has a solution. Observe that the spectrum of $s(n)$ is equal to zero on A.

Example: The method described above was used to generate a DTA signal from the sequence $x(n) = [1 \quad 2 \quad 1 \quad 2]$ and from the Daubechies scaling filter db16. We choose constants a and b such that the DTFT is equal to zero at $\omega = -2.4$ and $\omega = -\pi + 0.001$ for $x(n)$ and the filter respectively. The ω values are derived empirically. Results are compared with the DTA signals obtained using the algorithm in [2]. For both cases we illustrate the results using 256 point DFTs.

For $x(n) = [1 \quad 2 \quad 1 \quad 2]$, Figure (1) shows the magnitude of the spectra of the DTA signals generated by both methods. As formulated, the spectrum by the new method is equal to zero at $\omega = -2.4$, and is small in the neighborhood of $\omega = -2.4$. This is not the case for the one generated by the algorithm in [2]. Note that the magnetude of

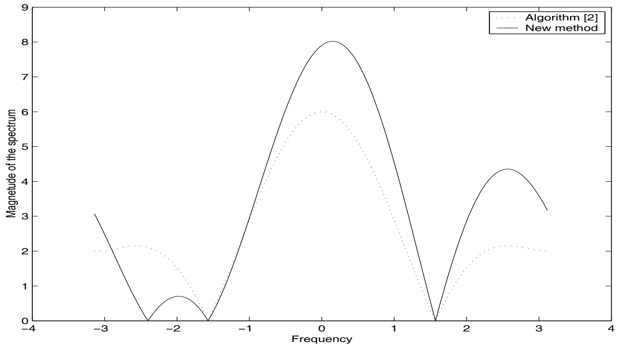

Figure 1: Discrete analytic signal for $x(n) = [1 \quad 2 \quad 1 \quad 2]$.

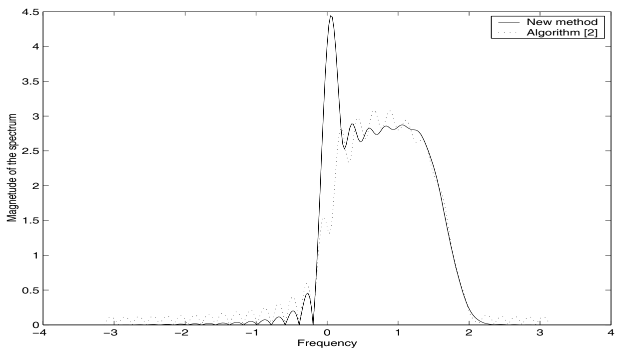

Figure 2: Discrete analytic signal for Daubechies scaling filter of length 32.

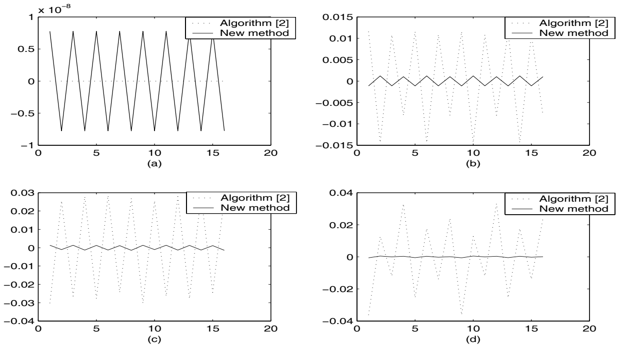

Figure 3: Subband energy for both transform. (a) Level-1 bandpass subband energy,(b) Level-2 bandpass subband energy,(c) Level-3 bandpass subband energy,(d) Level-3 lowpass subband energy.

the spectrum obtained by the algorithm in [2] is *symmetric*. For the db16 filter, Figure (2) shows that the magnitude of the spectrum of the DTA signal vanishes faster than that for the spectrum generated by the algorithm in [2].

4. EXPERIMENTAL RESULTS

The degree of aliasing in a wavelet decomposition is determined by a measure of the shiftability of the coefficients. A transform is defined as shiftable [5] when the coefficient energy in each subband is conserved under input-signal shifts. DTA signals corresponding to the db16 scaling filter were generated by the two methods and utilized as the scaling function filters. Quadrature mirror filters associated with each of the two filters served as wavelet filters for the two methods. Subband energy for both transforms was determined over 16 circular shifts of an impulse that was fed to the analytic scaling and wavelet filters . Figure (3) shows the transform subband energies at the different scales, as a function of input signal shifts.

We observe large oscillations in the subband-energy corresponding to the algorithm [2] in (b), (c), (d) whereas it is almost constant for the new method. The subband energy in (a) oscillates, but the variation is of order 10^{-8}. We conclude that the new method generates a spectrum that suppresses more negative frequencies than that obtained by using algorithm [2]. Accordingly, aliasing is considerably reduced. Both methods retain the original signal as the real part of the analytic signal. However, orthogonality of the real and imaginary parts is not maintained by the new method.

5. CONCLUSION

We have proposed a new method for generating a DTA signal for which the algorithm in [2] is a special case $(a = b = 0)$. The advantage of the method is that it assures better suppression of negative frequencies. Beside zeroing the DTFT of the DTA signal at $\omega_k = 2\pi k/N$, where $N/2 + 1 \leq k \leq N - 1$, it also zeros the DTFT of the DTA signal at a point in the negative frequency range, thus leading to improved shiftability.

6. REFERENCES

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