POLYSPECTRA OF ANALYTIC SIGNALS

Peter J. Schreier
Electrical & Computer Engineering
Univ. of Colorado, Boulder, CO 80309-0425
www.peter-schreier.com
peter@peter-schreier.com

Louis L. Scharf
Electrical & Computer Eng. and Statistics
Colorado State University
Ft. Collins, CO 80523
scharf@engr.colostate.edu

ABSTRACT

For complex signals, n-th order moment functions can be defined in 2^n different ways, depending on the placement of complex conjugates. We demonstrate that, for stationary analytic signals, only a few of these different moments are actually required for a complete n-th order description. Which, and how many of them, depends on the signal's spectrum. We investigate properties of n-th order moments and spectra with different conjugation patterns and show how they provide different information about the signal.

1. INTRODUCTION

The reason why most papers in higher-order spectrum analysis develop results for real rather than complex signals is that the complex case is so much more complicated: There are 2^n different n-th order moment functions, depending on where complex conjugate operators are placed. Papers that deal with complex signals often consider only one of these possible definitions, which is usually one with $\lfloor \frac{n}{2} \rfloor$ conjugates (where $\lfloor \cdot \rfloor$ denotes the floor function). This approach may be valid for some special cases, but not for others.

For instance, in the second-order case, the standard covariance $Es(t_1)s^*(t_2)$ must in general be complemented by the complementary covariance $Es(t_1)s(t_2)$. Only if $s(t)$ is proper will the complementary covariance vanish and thus not carry information. Propriety simplifies second-order analysis significantly, and it is often implicitly assumed. Picinbono [3] has extended the definition of propriety to higher orders. A process is called *n-th order proper* if its only non-zero moments up to order n have an equal number of conjugated and non-conjugated terms. The rationale behind this is that, under this definition, a proper Gaussian process is n-th order proper for all n.

While we cannot expect the proper case in general, this does not mean that we have to consider all 2^n conjugation patterns for an n-th order moment function. Obviously, all moment functions with q and $n-q$ conjugates are equivalent because they are related to each other through complex conjugation and/or coordinate transformations. Also, it is possible to have a number of moment functions equal to zero, even when the signal is not n-th order proper. In this paper, we investigate exactly which n-th order moment functions are required for a complete n-th order characterization of *stationary analytic* signals.

2. HIGHER-ORDER SPECTRA OF REAL SIGNALS

To lay the foundation for the discussion of higher-order spectra of analytic signals, we need to talk first about higher-order spectra of real signals. Higher-order moments can be defined as time-averages or as ensemble-averages. Obviously, ensemble-averaging is only possible when there is a way to obtain multiple realizations of a random process. Time-averaging, on the other hand, can be performed for both deterministic signals and individual sample paths of random processes.

Time-Averages: Let $x(t)$ be a real zero-mean continuous-time signal, either deterministic or a sample path of a random process. We define its n-th order moment function as the time-average of length T,

$$k^c_{x\cdots x}(\tau_1,\ldots,\tau_{n-1}) = \langle x(t)x(t+\tau_1)\cdots x(t+\tau_{n-1})\rangle \quad (1)$$
$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau_1)\cdots x(t+\tau_{n-1})\,dt$$

The superscript c indicates that $k^c_{x\cdots x}(\tau_1,\ldots,\tau_{n-1})$ is the moment function of a *continuous*-time signal. Let $X(f)$ be the Fourier transform of $x(t)$ windowed by a rectangular window of length T. The n-th order polyspectrum is then

$$K^c_{x\cdots x}(f_1,\ldots,f_{n-1}) = \tfrac{1}{T}X(f_1)\cdots X(f_{n-1})X^*(f_1+\ldots+f_{n-1}).$$

Sampling $x(t)$ at rate $f_s = 1/T_s$ yields the discrete-time signal $x[k] = x(kT_s)$. The n-th order moment function of $x[k]$ is the time-average of length N

$$k^d_{x\cdots x}[\kappa_1,\ldots,\kappa_{n-1}] = \langle x[k]x[k+\kappa_1]\cdots x[k+\kappa_{n-1}]\rangle$$
$$= \sum_{k=-N/2}^{N/2} x[k]x[k+\kappa_1]\cdots x[k+\kappa_{n-1}],$$

where the superscript d indicates discrete-time. Its $(n-1)$-dimensional discrete-time Fourier transform yields the corresponding n-th order polyspectrum $K^d_{x\cdots x}(f_1,\ldots,f_{n-1})$. The

0-7803-8484-9/04/$20.00 ©2004 IEEE

moment function of the sampled signal $x[k]$ is in general not equal to a sampled version of the moment function of $x(t)$:

$$k^d_{x\cdots x}[\kappa_1,...,\kappa_{n-1}] \neq k^c_{x\cdots x}(\kappa_1 T_s,...,\kappa_{n-1}T_s) \qquad (2)$$

The reason is that time-averaging and sampling are operations that do not commute. On the left hand side of (2), we sample first and then compute the time-average, whereas on the right hand side, we average first and then sample the time-average. It has been shown by Pflug *et al.* [1] that equality in (2) does hold if the sampling frequency f_s is chosen as $f_s \geq n f_{max}$, where f_{max} denotes the least upper bound of frequencies where $X(f)$ is nonzero.

Ensemble-Averages: Now assume that $x(t)$ is a real zero-mean continuous-time *random* signal. We also require that $x(t)$ be *m-th order harmonizable*, which means it can be expressed as

$$x(t) \overset{a.s.}{=} \int_{-\infty}^{\infty} e^{j2\pi ft} d\xi(f),$$

where $\xi(f)$ is an m-th order random function. Its increments $d\xi(f)$ satisfy the Hermitian symmetry $d\xi(f) = d\xi^*(-f)$ and have moments defined up to m-th order:

$$E\, d\xi(f_1)\cdots d\xi(f_n) = d^n\Phi_{x\cdots x}(f_1,...,f_n), \quad n = 1,...,m.$$

Since $x(t)$ is harmonizable, so are all of its moment functions, $n = 1,...,m$:

$$r^c_{x\cdots x}(t,\tau_1,...,\tau_{n-1}) = Ex(t)x(t+\tau_1)\cdots x(t+\tau_{n-1}) \qquad (3)$$

$$= \int\cdots\int_{-\infty}^{\infty} \exp\left(j2\pi\left[\sum_{i=1}^{n-1} f_i\tau_i + \sum_{i=1}^{n} f_i t\right]\right) d^n\Phi_{x\cdots x}(f_1,...,f_n)$$

Now, if $\Phi_{x\cdots x}(f_1,...,f_n)$ is absolutely continuous, there exists a density, the spectral correlation

$$R^c_{x\cdots x}(f_1,...,f_n) = \frac{\partial^n \Phi_{x\cdots x}(f_1,...,f_n)}{\partial f_1\cdots\partial f_n}.$$

We will allow the use of delta functions, and therefore, we can define $R^c_{x\cdots x}(f_1,...,f_n)$ even in the absence of absolute continuity of $\Phi_{x\cdots x}(f_1,...,f_n)$. While we have to keep in mind that this can make $R^c_{x\cdots x}(f_1,...,f_n)$ unbounded, integrals with respect to $R^c_{x\cdots x}(f_1,...,f_n)\, df_1\cdots df_n$ then work like integrals with respect to $d^n\Phi_{x\cdots x}(f_1,...,f_n)$.

Suppose now that $x(t)$ is stationary up to order m, which says that its n-th order correlation, $n = 1,...,m$, does not depend on t. From (3) it is evident that, in the stationary case, $R^c_{x\cdots x}(f_1,...,f_n)$ can only be non-zero on the *stationary manifold* $f_1 + ... + f_n = 0$. Then (3) is a simple Fourier transform relationship between the *stationary moment* $m^c_{x\cdots x}(\tau_1,...,\tau_{n-1}) = r^c_{x\cdots x}(0,\tau_1,...,\tau_{n-1})$ and the *polyspectrum* $M^c_{x\cdots x}(f_1,...,f_{n-1})$, which are both functions of $n-1$ variables only:

$$m^c_{x\cdots x}(\tau_1,...,\tau_{n-1}) = \int\cdots\int_{-\infty}^{\infty} M^c_{x\cdots x}(f_1,...,f_{n-1}) \times e^{j2\pi\sum_{i=1}^{n-1} f_i\tau_i}\, df_1\cdots df_{n-1}.$$

If we sample the random process $x(t)$ to obtain $x[k] = x(kT_s)$, the n-th order correlation function of $x[k]$ is

$$r^d_{x\cdots x}[k,\kappa_1,...,\kappa_{n-1}] = Ex[k]x[k+\kappa_1]\cdots x[k+\kappa_{n-1}] \qquad (4)$$

$$= Ex(kT_s)x(kT_s+\kappa_1 T_s)\cdots x(kT_s+\kappa_{n-1}T_s)$$

$$= r^c_{x\cdots x}(kT_s,\kappa_1 T_s,...,\kappa_{n-1}T_s), \qquad (5)$$

which is a sampled version of the n-th order correlation function of $x(t)$. This result holds regardless of the sampling frequency. It is due to the fact that *ensemble*-averaging and sampling are operations that do commute, unlike *time*-averaging and sampling. Thus, the equality (4)=(5) stands in contrast to the time-average case, where (2) is an inequality unless $f_s \geq nf_{max}$. Differences between higher-order moments defined as time-averages and those defined as ensemble-averages are further illuminated in [4].

3. ANALYTIC SIGNALS

In this section, we look at time- and ensemble-averaged polyspectra for deterministic and *stationary* stochastic analytic signals. The analytic signal is arguably the most important example of a complex signal. It is constructed from the real signal $x(t)$ as $s(t) = x(t) + j\hat{x}(t)$, where $\hat{x}(t)$ denotes the Hilbert transform of $x(t)$. Thus, we have $S(f) = u(f)X(f)$ for sample paths and $d\sigma(f) = u(f)d\xi(f)$ for stochastic processes, where $u(f)$ denotes twice the Heaviside step function. It might be tempting to conclude from this that polyspectra of analytic signals can only be nonzero for $f_i \geq 0$, $i = 1,...,n-1$. This is not true. Using $S(f) = u(f)X(f)$, we find that the n-th order polyspectra of the analytic realization $s(t)$ are connected to the n-th order polyspectrum of its corresponding real signal $x(t)$ as

$$K^c_{s^{\square_n}s^{\square_1}\cdots s^{\square_{n-1}}}(f_1,...,f_{n-1}) = u(\pm_1 f_1)\cdots u(\pm_{n-1}f_{n-1}) \times u(\pm_n(-(f_1+...+f_{n-1})))K^c_{x\cdots x}(f_1,...,f_{n-1}), \qquad (6)$$

where $\square_i$ stands for either $+1$ or the conjugating star $*$, and $\pm_i$ is $+1$ for $\square_i = +1$ and -1 for $\square_i = *$. The same relationship as in (6) holds for $M^c_{s^{\square_n}s^{\square_1}\cdots s^{\square_{n-1}}}(f_1,...,f_{n-1})$ and the following discussion applies to ensemble-averages of random processes, as well. We can see from (6) that $K^c_{s^{\square_n}s^{\square_1}\cdots s^{\square_{n-1}}}(f_1,...,f_{n-1})$ cuts out regions of the polyspectrum of the real signal, $K^c_{x\cdots x}(f_1,...,f_{n-1})$. Therefore, on its nonzero domain, $K^c_{s^{\square_n}s^{\square_1}\cdots s^{\square_{n-1}}}(f_1,...,f_{n-1})$ also inherits the symmetries of $K^c_{x\cdots x}(f_1,...,f_{n-1})$. The conjugation pattern determines the selected region and thus also the symmetry properties. Only the pattern $s^*ss\cdots s$ selects the all-positive orthant $f_i > 0$, $i = 1,...,n-1$.

Conditions for Nonzero Polyspectra: Polyspectra of order n for analytic signals can be zero depending on the number of conjugates, q, and the spectrum of $s(t)$. Let f_{min} and f_{max} denote the minimum and maximum frequency where $S(f)$ is nonzero. In order to obtain a nonzero polyspectrum $K^{c}_{s^{\square_n} s^{\square_1} \dots s^{\square_{n-1}}}(f_1, \dots, f_{n-1})$, there must be overlap between the support of $S^{\square_1}(\pm_1 f_1) \cdots S^{\square_{n-1}}(\pm_{n-1} f_{n-1})$ and the support of $S^{\square_n}(\pm_n(-f_1 - \dots - f_{n-1}))$. The lowest nonzero frequency of $S^{\square_i}(\pm_i f_i)$, $i = 1, \dots, n-1$, is f_{min} if $\square_i = +1$ and $-f_{max}$ if $\square_i = *$, and similarly, the highest nonzero frequency is f_{max} if $\square_i = +1$ and $-f_{min}$ if $\square_i = *$. Take first the case where $q \geq 1$, so we can assume without loss of generality that $\square_n = *$. Then we obtain the required overlap if both of these conditions hold:

$$(n-q) f_{min} - (q-1) f_{max} < f_{max}, \tag{7}$$
$$(n-q) f_{max} - (q-1) f_{min} > f_{min}. \tag{8}$$

Now, if $q = 0$, then $\square_n = +1$ and we require

$$(n-1) f_{min} < -f_{min},$$
$$(n-1) f_{max} > -f_{max},$$

which shows that (7) and (8) are valid for $q = 0$, as well. Since one of the inequalities (7) and (8) will always be trivially satisfied, a simple condition to guarantee a nonzero $K^{c}_{s^{\square_n} s^{\square_1} \dots s^{\square_{n-1}}}(f_1, \dots, f_{n-1})$ is

$$\begin{aligned} (n-q) f_{min} - q f_{max} &< 0, \quad 2q \leq n \\ (n-q) f_{max} - q f_{min} &> 0, \quad 2q > n. \end{aligned} \tag{9}$$

This result holds for polyspectra defined through time- or ensemble-averages. It bears some comments. First of all, if either $q = 0$ or $q = n$, the condition (9) is $f_{min} < 0$, which is impossible for an analytic signal. Thus, no-starring and all-starring conjugation patterns yield zero n-th order polyspectrum. On the other hand, if n is even, moments with $n/2$ stars will always be nonzero.

Let us now investigate what effect sampling the band-limited signal $s(t)$ with $f_s \geq 2 f_{max}$ has on its polyspectra. For sampled realizations, there can be additional nonzero time-averaged polyspectra with other starring patterns. We get a nonzero *aliased* domain in $K^{d}_{s^{\square_n} s^{\square_1} \dots s^{\square_{n-1}}}(f_1, \dots, f_{n-1})$ if there exists overlap between the support of the hypercube described by $S^{\square_1}(\pm_1 f_1) \cdots S^{\square_{n-1}}(\pm_{n-1} f_{n-1})$ and the support of either $S^{\square_n}(\pm_n f_n + f_s)$ or $S^{\square_n}(\pm_n f_n - f_s)$. Repeating the reasoning from above, this translates into the condition

$$\begin{aligned} (n-q) f_{min} - q f_{max} < f_s < (n-q) f_{max} - q f_{min}, \quad 2q \leq n, \\ (n-q) f_{min} - q f_{max} < -f_s < (n-q) f_{max} - q f_{min}, \quad 2q > n. \end{aligned}$$

for a nonzero aliased domain in $K^{d}_{s^{\square_n} s^{\square_1} \dots s^{\square_{n-1}}}(f_1, \dots, f_{n-1})$. The condition (9) for a nonzero unaliased domain remains valid.

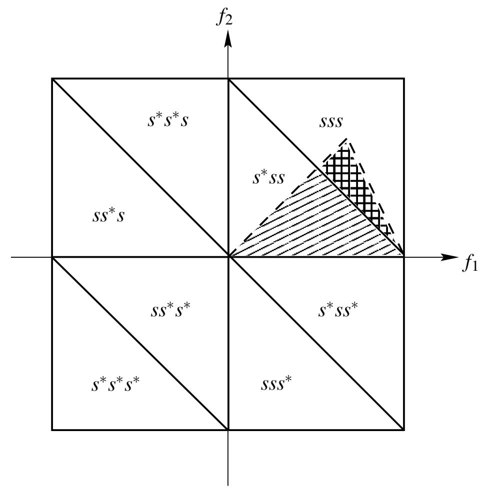

Fig. 1. Support of bispectra $K^{d}_{s^{\square_3} s^{\square_1} s^{\square_2}}(f_1, f_2)$ with different starring patterns for analytic $s(t)$ sampled with $f_s \geq 2 f_{max}$. The principal unaliased domain of $K^{d}_{xxx}(f_1, f_2)$ is diagonally hatched, the principal aliased domain of $K^{d}_{xxx}(f_1, f_2)$ is cross-hatched.

Because ensemble-averaging and sampling are operations that commute, there is no higher-order aliasing for ensemble-averages of stochastic processes [4]. That is, an ensemble-averaged polyspectrum will always be zero on the aliased domain of time-averaged polyspectra, whether the signals are continuous or sampled.

Bi- and Trispectrum: We first illustrate the results in the preceding section with the bispectrum. While $K^{c}_{sss}(f_1, f_2)$ and $K^{c}_{s^*s^*s^*}(f_1, f_2)$ must be identically zero, $K^{d}_{sss}(f_1, f_2)$ and $K^{d}_{s^*s^*s^*}(f_1, f_2)$ can be nonzero if $f_{min} < 1/3 f_s$ and $f_{max} > 1/3 f_s$. The advantage that a complex analytic description has is that higher-order aliasing takes place in $K^{d}_{sss}(f_1, f_2)$ and $K^{d}_{s^*s^*s^*}(f_1, f_2)$, and there only. I.e., $K^{d}_{s^{\square_3} s^{\square_1} s^{\square_2}}(f_1, f_2)$ with $q = 1, 2$ is unaffected by higher-order aliasing. This also implies that for $q = 1, 2$, contrary to (2),

$$k^{d}_{s^{\square_3} s^{\square_1} s^{\square_2}}[\kappa_1, \dots, \kappa_{n-1}] = k^{c}_{s^{\square_3} s^{\square_1} s^{\square_2}}(\kappa_1 T_s, \dots, \kappa_{n-1} T_s),$$

is an equality, as long as the signal is not undersampled, i.e., $f_s \geq 2 f_{max}$. Figure 1 depicts how the support of bispectra with different starring patterns covers the frequency plane $-f_N \leq f_1, f_2 \leq f_N$. We can see how $K^{d}_{s^*ss}(f_1, f_2)$ covers the principal unaliased domain and $K^{d}_{sss}(f_1, f_2)$ covers the principal aliased domain of $K^{d}_{xxx}(f_1, f_2)$ in their most common definitions. This confirms the notion that s^*ss and sss are the canonical starring patterns for the unaliased and aliased domains.

If we evaluate the conditions (9) and (3) for nonzero unaliased and aliased domains in trispectra with different numbers of conjugation operations, we obtain Table 1. There are two types of unaliased domains, type 1 described by

q	unaliased domains	aliased domains
$0,4$	always zero	$4f_{min} < f_s < 4f_{max}$ (type 2)
$1,3$	$3f_{min} < f_{max}$ (type 1)	$3f_{max} - f_{min} > f_s$ (type 1)
2	always nonzero (type 2)	always zero

Table 1. Necessary conditions for nonzero unaliased and aliased domains in trispectra with different number of conjugates for an analytic signal sampled with $f_s \geq 2f_{max}$.

$q = 1,3$, type 2 by $q = 2$, and two types of aliased domains, type 1 described by $q = 1,3$, type 2 by $q = 0,4$. Let us first talk about the two types of unaliased domains. Type 1 only exists (i.e., is nonzero) if $3f_{min} < f_{max}$, whereas type 2 always exists. Only type 1 has the ability to distinguish between two signals $s(t)$ and $r(t)$ that differ only by a constant phase shift. If $r(t) = s(t)e^{j\phi}$, then $k^c_{r^*rrr}(\tau_1,\tau_2,\tau_3) = k^c_{s^*sss}(\tau_1,\tau_2,\tau_3)e^{2j\phi}$, whereas $k^c_{r^*r^*rr}(\tau_1,\tau_2,\tau_3)$ is identical to $k^c_{s^*s^*ss}(\tau_1,\tau_2,\tau_3)$. This means that, if $S(f)$ does not satisfy $3f_{min} < f_{max}$, $s(t)$ and $r(t)$ will be indistinguishable in trispectral analysis. Thus, they will also have the same kurtosis. This is important in deconvolution techniques [2, 5].

At this point, it is important to mention that the usual definition given for the kurtosis γ^4 of an analytic signal, $\langle |s(t)|^4 \rangle$ or $E\,|s(t)|^4$, a naive extension of the definition for a real signal, is incorrect because it only considers contributions from unaliased domains of type 2. The correct result (for time-averages) is

$$\gamma^4 = \langle x^4(t) \rangle = \left\langle \left(\tfrac{1}{2}(s(t)+s^*(t))\right)^4 \right\rangle = \tfrac{1}{2}\mathrm{Re}\,\langle s^3(t)s^*(t)) \rangle + \tfrac{3}{8}\langle |s(t)|^4 \rangle$$

which takes into account the contribution from unaliased domains of type 1. We can evaluate $\langle s^3(t)s^*(t) \rangle$ by integrating $K^c_{s^{\square_4}s^{\square_1}s^{\square_2}s^{\square_3}}(f_1,f_2,f_3)$ for one particular conjugation pattern with $q = 1$ or $q = 3$ over its nonzero domain, and similarly $\langle |s(t)|^4 \rangle$ by integrating the trispectrum for one particular conjugation pattern with $q = 2$. In the case of a sampled realization, the kurtosis is

$$\gamma^4 = \tfrac{1}{8}\mathrm{Re}\,\langle s^4[k] \rangle + \tfrac{1}{2}\mathrm{Re}\,\langle s^3[k]s^*[k]) \rangle + \tfrac{3}{8}\langle |s[k]|^4 \rangle,$$

which considers the additional contribution from aliased domains of type 2. Note that while $\langle |s(t)|^4 \rangle = \langle |s[k]|^4 \rangle$, we have $\langle s^3(t)s^*(t)) \rangle \neq \langle s^3[k]s^*[k]) \rangle$ if $3f_{max} - f_{min} > f_s$ because of the existence of aliased domains of type 1. An aliased domain of type 1 can exist even if type 2 does not, and vice versa. Both types will disappear if the signal is sufficiently oversampled with $f_s \geq 4f_{max}$. Aliased domains of both types are always zero for ensemble-averaged trispectra of stochastic processes sampled with $f_s \geq 2f_{max}$.

Beyond the Trispectrum: We now generalize the discussion of the preceding section to n-th order polyspectra. Let us first talk about the existence condition (9) for nonzero unaliased domains of time- or ensemble-averaged polyspectra. We can assume without loss of generality that $2q \leq n$ because polyspectra with q and $n-q$ conjugates are equivalent. For fixed order n, satisfaction of (9) for $q-1$ implies satisfaction for q. Thus, if unaliased domains with $q-1$ conjugates are nonzero, so will be unaliased domains with q conjugates. The reverse conclusion is that if unaliased domains with q stars are zero, so will be unaliased domains with $q-1$ stars.

The corresponding discussion for the existence condition (3) for nonzero aliased domains of time-averaged polyspectra is more complicated because there are two inequalities involved, which behave contrarily. For fixed order n, satisfaction of the *left* inequality of (3) for $q-1$ implies satisfaction for q. For the *right* inequality, it goes the opposite direction: If it holds for q, it will also hold for $q-1$. Since both inequalities must be fulfilled for a nonzero aliased domain, we conclude that if the *left* inequality is violated for q, aliased domains with q or fewer stars will be zero. If the *right* inequality is violated for q, aliased domains with q or more stars will be zero. For instance, sampling the signal with $f_s > nf_{max}$ violates the right inequality for $q = 0$, and therefore guarantees that there will be no higher-order aliasing at all. Note that there can also be inequalities that are automatically fulfilled. For example, for $n = 4$, $q = 1$ the left inequality is $3f_{min} - f_{max} < f_s$, which is always true because $f_{min} < f_{max} \leq 2f_s$.

4. CONCLUSIONS

Higher-order analysis of complex signals is easy in the n-th order proper case, where there is at most one distinct n-th order moment to be considered. This situation, however, is far from being general. In some problems, such as deconvolution, the n-th order proper case is even detrimental because it does not allow detection of constant phase shifts. In this paper, we have shed some light on the improper case. For stationary analytic signals, we have investigated properties of n-th order moments and spectra with different conjugation configurations. Which moments are nonzero depends on the spectrum of the signal.

5. REFERENCES

[1] L. A. Pflug, G. E. Ioup, J. W. Ioup, "Sampling requirements and aliasing for higher-order correlations," *J. Acoust. Soc. Am.* **94**, 2159 – 2172 (1993).

[2] L. A. Pflug, "Principal domains of the trispectrum, signal bandwidth, and implications for deconvolution," *Geophysics* **65**, 958 – 969 (2000).

[3] B. Picinbono, "On circularity," *IEEE Trans. Signal Processing* **42**, 3473 – 3482 (1994).

[4] P. J. Schreier, L. L. Scharf, "Polyspectra of complex signals," submitted to *IEEE Trans. Signal Processing*

[5] A. T. Walden, M. L. Williams, "Deconvolution, bandwidth, and the trispectrum," *J. Am. Stat. Assoc.* **88**, 1323 – 1329 (1993).