DESIGN OF COMPLEX ALLPASS FILTERS

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ABSTRACT

This paper presents the design of complex allpass filters that satisfy a desired degree of flatness and a desired phase at any specified frequency point. The set of linear equations are derived based on the specifications of the flatness and the values of the phase at the given frequency points. The filter coefficients are obtained by solving this set of equations.

1. INTRODUCTION

This paper treats the design of complex allpass filters with given degrees of flatness at prescribed frequency points.

The complex allpass filters are used in the design of even degree IIR digital filters [1], [2]. The characteristics of designed IIR filter are dependent on a degree of flatness at specific frequency points of the complex allpass filter, and on the approximation of its phase. The problem of the allpass filter phase approximation is treated in [1], [3], [4], [5], [6]. The design of a real allpass filter with specified degree of flatness at the frequency points $\omega = 0$ and $\omega = \pi$ is proposed in [3].

The main idea of this paper is to generalize the method [3] for the design of complex allpass filters having the desired degree of flatness at any prescribed frequency point.

The paper is organized as follows. The equations for maximally flat group delay of an all pole filter are derived in Section 2. The design of complex allpass filters based on these equations is presented in Section 3. The method is illustrated with two examples.

2. MAXIMALLY FLAT ALL POLE FILTER

Consider an all pole filter given by

$$D(z) = \frac{1}{F(z)}, \tag{1}$$

where

$$F(z) = \sum_{n=0}^{N} f_n z^{-n}, \tag{2}$$

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and f_n are complex coefficients, i.e. $f_n = r_n e^{j\phi_n}$, where r_n is the amplitude and ϕ_n is the phase of f_n. Coefficients f_n can also be expressed as $f_n = f_{\mathrm{R}n} + j f_{\mathrm{I}n}$, where $f_{\mathrm{R}n}$ and $f_{\mathrm{I}n}$ are the real and the imaginary part of f_n, respectively.

The Fourier transform of f_n, $n = 0 \dots N$, $F(e^{j\omega})$ is given by

$$F(e^{j\omega}) = \sum_{n=0}^{N} f_n e^{-j\omega n} \tag{3a}$$

$$= \sum_{n=0}^{N} \cos(\omega n - \phi_n) r_n - j \sum_{n=0}^{N} \sin(\omega n - \phi_n) r_n. \tag{3b}$$

The phases of $D(e^{j\omega})$ and $F(e^{j\omega})$ are related as

$$\phi_D(\omega) = -\phi_F(\omega). \tag{4}$$

The group delay is the negative derivative of the phase, given by

$$G(\omega) = -\frac{d}{d\omega}\{\phi_D(\omega)\} = \frac{d}{d\omega}\{\phi_F(\omega)\}. \tag{5}$$

The conditions for maximally flat group delay are as follows,

$$G(\omega) = \tau \tag{6a}$$

$$G^{(k)}(\omega) = 0, \qquad k = 1 \dots K \tag{6b}$$

where τ is the desired group delay, $G^{(k)}(\omega)$ indicates the k^{th} derivative of $G(\omega)$, and K is an integer.

Using (3b) and (5), the negative derivative of the phase $\phi_D(\omega)$ can be written as

$$-\frac{d\phi_D(\omega)}{d\omega} = -\frac{d}{d\omega}\left\{\tan^{-1}\left(\frac{\sum_{n=0}^{N}\sin(\omega n - \phi_n) r_n}{\sum_{n=0}^{N}\cos(\omega n - \phi_n) r_n}\right)\right\}. \tag{7}$$

By performing some simple trigonometric transformations, we arrive at,

$$\frac{d}{d\omega}\left\{\sum_{n=0}^{N} r(n)\sin(\omega n - \phi_D(\omega) - \phi(n))\right\}$$
$$= \sum_{n=0}^{N} r_n \cos(\omega n - \phi_D(\omega) - \phi_n)\left(n - \frac{d\phi_D}{d\omega}\right) = 0. \quad (8)$$

Using (5) and the condition (6a) it follows that,

$$\sum_{n=0}^{N} r_n \cos(\omega n - \phi_D(\omega) - \phi_n)(n+\tau) = 0. \quad (9)$$

From the condition (6b), for $k = 1$, we have that,

$$\frac{d^2}{d\omega^2}\left\{\sum_{n=0}^{N} r(n)\sin(\omega n - \phi_D(\omega) - \phi(n))\right\}$$
$$= -\sum_{n=0}^{N} r_n \sin(\omega n - \phi_D(\omega) - \phi_n)\left(n - \frac{d\phi_D}{d\omega}\right)^2$$
$$- \sum_{n=0}^{N} r_n \cos(\omega n - \phi_D(\omega) - \phi_n)\frac{d^2\phi_D}{d\omega^2}$$
$$= \sum_{n=0}^{N} r_n \sin(\omega n - \phi_D(\omega) - \phi_n)(n+\tau)^2 = 0. \quad (10)$$

By continuing in the same way for all values of k, we obtain the following set of equations,

$$\sum_{n=0}^{N} (n+\tau)^k \cos(\omega n - \phi_D(\omega) - \phi_n) r_n = 0, \qquad k \text{ odd}, \quad (11a)$$

$$\sum_{n=0}^{N} (n+\tau)^k \sin(\omega n - \phi_D(\omega) - \phi_n) r_n = 0, \qquad k \text{ even}, \quad (11b)$$

where $k = 0 \ldots K+1$. By setting $k = 0$ and $k = 1$, the equations for the desired phase, $\phi_D(\omega)$, and the desired group delay, τ follow. The remaining K equations satisfy the condition (6b).

Using $f_0 = r_0 e^{j\phi_0} = 1$ the equations (11) can be written as,

$$\sum_{n=1}^{N} (n+\tau)^k \cos(\omega n - \phi_D(\omega) - \phi_n) r_n = -\tau^k \cos(\phi_D(\omega)), \qquad k \text{ odd}, \quad (12a)$$

$$\sum_{n=1}^{N} (n+\tau)^k \sin(\omega n - \phi_D(\omega) - \phi_n) r_n = \tau^k \sin(\phi_D(\omega)), \qquad k \text{ even}, \quad (12b)$$

or

$$\sum_{n=1}^{N} \left\{(n+\tau)^k \cos(\omega n - \phi_D(\omega))\right\} f_{\mathrm{R}n}$$
$$+ \sum_{n=1}^{N} \left\{(n+\tau)^k \sin(\omega n - \phi_D(\omega))\right\} f_{\mathrm{I}n}$$
$$= -\tau^k \cos(\phi_D(\omega)), \qquad k \text{ odd}, \quad (13a)$$

$$\sum_{n=1}^{N} \left\{(n+\tau)^k \sin(\omega n - \phi_D(\omega))\right\} f_{\mathrm{R}n}$$
$$- \sum_{n=1}^{N} \left\{(n+\tau)^k \cos(\omega n - \phi_D(\omega))\right\} f_{\mathrm{I}n}$$
$$= \tau^k \sin(\phi_D(\omega)), \qquad k \text{ even}. \quad (13b)$$

Equations (13) are the general equations for the maximally flat group delay at any frequency point. The solution of this set of equations are the coefficients of the complex allpass filter.

One special case of (13) is obtained for $f_{\mathrm{I}n} = 0$, $\omega = 0$ and $\omega = \pi$. In this case, the phase $\phi_D(\omega)$ can be 0 or π depending on the sign of $\sum_{n=0}^{N} f_{\mathrm{R}n}$. This result is presented in [3] in the form

$$\sum_{n=1}^{N} (n+\tau)^{2k+1} f_{\mathrm{R}n} = -\tau^{2k+1}, \quad (14)$$

$$\sum_{n=1}^{N} (-1)^n (n+\tau)^{2k+1} f_{\mathrm{R}n} = -\tau^{2k+1}. \quad (15)$$

3. COMPLEX ALLPASS FILTER

Consider a complex allpass filter $A(z)$ in the form

$$A(z) = z^{-N}\frac{\widetilde{F}(z)}{F(z)} = z^{-N}\frac{D(z)}{\widetilde{D}(z)}. \quad (16)$$

Here $\widetilde{F}(z)$ is the result of first conjugating the coefficients of z in the function $F(z)$, and then replacing z with z^{-1}, [7]. Suppose that the group delay of $D(z)$ is the desired group delay τ discussed in Section 2. The group delay of the complex allpass filter, τ_A, is given by

$$\tau_A = N + 2\tau, \quad (17)$$

so that the desired group delay τ can be written as

$$\tau = \frac{\tau_A - N}{2}. \quad (18)$$

If $\tau_A < N$, the poles of $A(z)$ are outside of the unit circle [3].

The phase $\phi_A(\omega)$ of $A(z)$ can be expressed as

$$\phi_A(\omega) = -\omega N + 2\phi_D(\omega), \tag{19}$$

where the desired phase $\phi_D(\omega)$ is given by

$$\phi_D(\omega) = \frac{\phi_A(\omega) + \omega N}{2}. \tag{20}$$

When $\tau > 0$ the phase of $A(z)$ satisfies [8]:

- $\phi_A(2\pi) = \phi_A(0) - 2N\pi$,
- $\phi_A(\omega)$ exhibits monotonic decreasing behavior.

In the following two examples we illustrate the design of maximally flat group delay complex allpass filters using (13), (18) and (20).

Example 1: In this example we design the complex allpass filter with these characteristics:

At frequency point $\omega_0/\pi = 1/3$ the desired phase is $\phi_{A_0}/\pi = -4$, and the specified degree of flatness is $K_0 = 8$. Similarly, at the frequency points $\omega_1/\pi = 4/5$, and $\omega_2/\pi = 8/5$ the desired phases are $\phi_{A_1}/\pi = -10.5$, and $\phi_{A_2}/\pi = -20.5$, respectively. The corresponding degrees of flatness are the same, i.e. $K_1 = K_2 = 6$. The specified group delay is the same in all frequency points and is equal to $\tau_A = 14$.

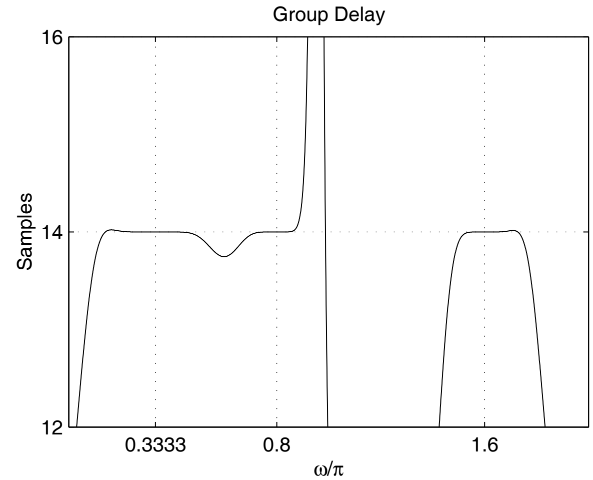
The number of coefficients N, is $(K_0+K_1+K_2+6)/2$. From (18) and (20) it follows that $\tau = 0.5$, $\phi_{D_0} = 0.5236$, $\phi_{D_1} = -0.1571$ and $\phi_{D_2} = 0.4712$. If we substitute these values into (13) we obtain a set of linear equations with 26 unknowns; 13 for $f_{\mathrm{R}n}$ and 13 for $f_{\mathrm{I}n}$, of the form

$$\mathbf{Af} = \mathbf{b} \tag{21}$$

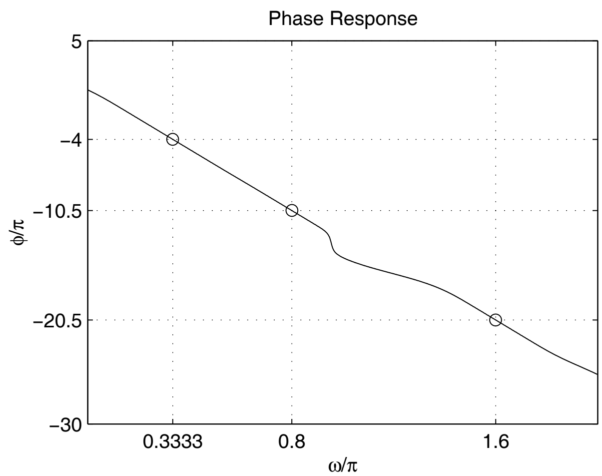
The first 10 rows of $\mathbf{A}$ correspond to the first frequency point ω_0, the next 8 rows correspond to the second frequency point ω_1, and the last 8 rows correspond to ω_2. The first 13 rows of the vector $\mathbf{f}$ are the values $f_{\mathrm{R}n}$, while the last 13 rows are the values of $f_{\mathrm{I}n}$. The entries in $\mathbf{b}$ are the right side in (13). Solving the set of equations (21), the coefficients of the complex allpass filter are computed and are listed in Table 1.

n	f_n	n	f_n
0	1.00000	7	–0.28938 + 0.09528j
1	0.09467 – 0.94300j	8	0.16238 + 0.03622j
2	–0.50693 + 0.44365j	9	–0.07009 – 0.08795j
3	0.84485 – 0.14725j	10	–0.00941 + 0.04869j
4	–0.55877 – 0.53642j	11	0.01119 – 0.01490j
5	0.10853 + 0.51413j	12	–0.00891 + 0.00128j
6	0.18520 – 0.42986j	13	0.00100 + 0.00167j

Table 1. Filter coefficients in Example 1



(a) Group Delay



(b) Phase Response

Fig. 1. Example 1

Fig. 1 illustrates the group delay and the phase of the designed allpass filter.

Example 2: In this example we design the complex allpass filter with the prescribed degree of flatness and phase at five frequency points, as follows: $\omega_0/\pi = 1/3$, $\omega_1/\pi = 3/5$, $\omega_2/\pi = 1$, $\omega_3/\pi = 3/2$ and $\omega_4/\pi = 9/5$. The degrees of flatness are $K_0 = 4$, $K_1 = 8$, $K_2 = 6$, $K_3 = 4$ and $K_4 = 8$, respectively, while the phases are $\phi_{A_0}/\pi = -6$, $\phi_{A_1}/\pi = -12.5$, $\phi_{A_2}/\pi = -20.5$, $\phi_{A_3}/\pi = -28.5$ and $\phi_{A_4}/\pi = -36.5$, respectively. The group delay in all frequency points is equal to $\tau_A = 24$.

From (18) and (20) we have: $\tau = 2$, $\phi_{D_0} = 1.0472$, $\phi_{D_1} = -0.7854$, $\phi_{D_2} = -0.7854$, $\phi_{D_3} = 2.3562$ and $\phi_{D_0} = -0.7854$. The coefficients of the complex allpass filter, which are listed in Table 2, result from solving the equations (13). Fig. 2 illustrates the group delay and the phase of the designed filter.

n	f_n	n	f_n
0	1.00000	11	0.52598 + 0.06816j
1	–0.32780 – 0.47823j	12	–0.49494 – 0.25491j
2	0.76126 + 1.04159j	13	–0.15125 + 0.43506j
3	1.16063 – 0.51237j	14	0.14680 – 0.16833j
4	–0.77454 + 0.14103j	15	–0.24035 – 0.03415j
5	1.13351 + 1.55667j	16	0.02809 + 0.08064j
6	0.51017 – 0.84977j	17	0.01145 – 0.07408j
7	–0.90543 + 0.96482j	18	–0.04378 + 0.00145j
8	0.96986 + 0.89534j	19	0.01513 + 0.00386j
9	–0.30326 – 0.65537j	20	–0.00190 – 0.00787j
10	–0.59011 + 0.98749j		

Table 2. Filter coefficients in Example 2

4. CONCLUSIONS

A new method for the design of complex allpass filters is presented. The designed filter satisfies the prescribed degree of flatness as well as the prescribed values of phases at any number of the frequency points. The filter coefficients are obtained by solving the set of linear equations. The proposed method can be useful for IIR filters design.

5. REFERENCES

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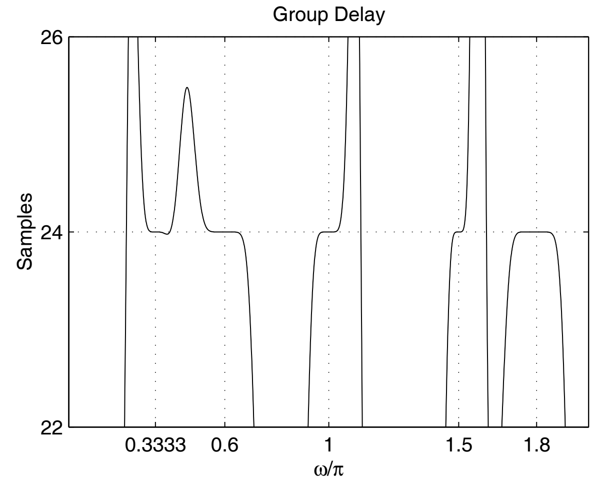
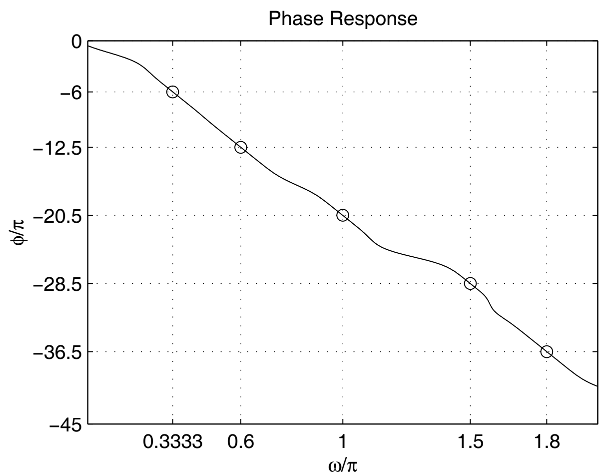


(a) Group Delay



(b) Phase Response

Fig. 2. Example 2

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