# RECURSIVE LEAST SQUARES SOLUTION TO SOURCE TRACKING USING TIME DIFFERENCE OF ARRIVAL

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**ABSTRACT**

A simple recursive tracking algorithm for moving sources is developed based on the time difference of arrival (TDOA) measurements. The proposed algorithm uses location estimates obtained from a stationary source localization algorithm and smoothes them according to a general motion model for the source. The motion model allows for source acceleration. A unified treatment of the iterative maximum likelihood and closed-form least squares algorithms for source localization by TDOA is presented. The ability of the proposed tracking algorithm to track maneuvering sources is demonstrated in simulation examples.

## 1. INTRODUCTION

The problem of passive tracking of moving sources is encountered in many diverse applications such as electronic warfare, surveillance, mobile user location in cellular communications and acoustic source localization in speech data acquisition systems. In passive tracking, the signal emitted by the source is utilized to determine the location and velocity of the source. One approach to passive tracking is to employ the time difference of arrival (TDOA) between signals received at multiple sensors (receivers). This approach leads to a set of nonlinear equations whose solution represents the intersection of multiple hyperbolae corresponding to TDOA measurements.

In this paper, we review the iterative maximum likelihood and closed-form least squares algorithms for source localization by TDOA. In particular, we provide a unified treatment of these algorithms in order to compare their pros and cons. We propose a simple recursive algorithm to track maneuvering sources by means of smoothing the source location estimates obtained from a constrained least squares solution. The proposed tracker uses a general motion model to fit the location estimates, resulting in initial location and velocity, and acceleration estimates. The recursive solution is implemented using the recursive least squares algorithm where the "autocorrelation" matrix does not need to be computed online. This provides a significant savings of the computational cost. The ability of the proposed tracker to smooth the location estimates, thereby improving the track estimate performance, is demonstrated in simulation examples.

The paper is organized as follows. Section 2 reviews the maximum likelihood location estimator and derives a Gauss-Newton solution for it. Closed-form least squares and iterative constrained least squares solutions are discussed in Section 3. Section 4 develops the proposed recursive tracking algorithm. Simulation examples are presented in Section 5.

This work was partially supported by a DSTO research agreement.

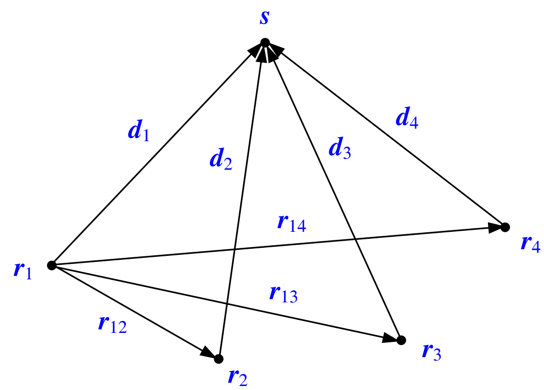


**Fig. 1**. Two-dimensional TDOA source localization geometry with four receivers.

## 2. MAXIMUM LIKELIHOOD LOCALIZATION BY TDOA

A two-dimensional source localization geometry using TDOA measurements from $N = 4$ receivers is shown in Fig. 1. The objective of source localization by TDOA is to determine the source location $\boldsymbol{s} = [x, y]^T$ (where $^T$ denotes the matrix transpose) by utilizing $N-1$ TDOA measurements obtained from $N \geq 3$ receivers at known locations $\boldsymbol{r}_i = [x_i, y_i]^T$, $i = 1, \ldots, N$.

TDOA between signals received at a pair of receivers is defined by

$$t_{ij} = t_j - t_i, \quad i, j \in \{1, \ldots, N\} \tag{1}$$

where $t_i$ is the time it takes for the signal transmitted by the source to arrive at the receiver $\boldsymbol{r}_i$, i.e.,

$$t_i = \frac{\|\boldsymbol{d}_i\|}{c}, \quad i \in \{1, \ldots, N\}. \tag{2}$$

Here $c$ is the speed of propagation for the transmitted signal, and $\boldsymbol{d}_i$ is the range vector for the receiver $\boldsymbol{r}_i$:

$$\boldsymbol{d}_i = \boldsymbol{s} - \boldsymbol{r}_i, \quad i \in \{1, \ldots, N\}. \tag{3}$$

Using (1) and (2), the range difference of arrival (RDOA), $g_{ij}$, can be defined as

$$g_{ij} = \|\boldsymbol{d}_j\| - \|\boldsymbol{d}_i\|, \quad i, j \in \{1, \dots, N\} \tag{4a}$$
$$= ct_{ij}. \tag{4b}$$

Each RDOA measurement defines a hyperbola of possible source locations. The source location is obtained from intersection of two or more hyperbolae by solving the following set of nonlinear equations for $\boldsymbol{s}$:

$$\begin{aligned} \|\boldsymbol{s} - \boldsymbol{r}_2\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| &= g_{12} \\ \|\boldsymbol{s} - \boldsymbol{r}_3\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| &= g_{13} \\ &\vdots \\ \|\boldsymbol{s} - \boldsymbol{r}_N\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| &= g_{1N}. \end{aligned} \tag{5}$$

In practice, we have to deal with noisy RDOA measurements $\tilde{g}_{1i}$ defined by

$$\tilde{g}_{1i} = g_{1i} + n_{1i}, \quad i = 2, \dots, N. \tag{6}$$

Here $n_{1i}$ is the RDOA noise which is assumed to be Gaussian. The RDOA noise covariance matrix is

$$\boldsymbol{\Sigma} = E\left\{ \begin{bmatrix} n_{12} \\ \vdots \\ n_{1N} \end{bmatrix} \begin{bmatrix} n_{12} & \cdots & n_{1N} \end{bmatrix} \right\}. \tag{7}$$

Supposing that the signal received at each receiver $\boldsymbol{r}_i$ is subject to i.i.d. additive noise and that TDOAs are estimated by generalized crosscorrelation [1], the covariance matrix can be shown to take the form [2]

$$\boldsymbol{\Sigma} = \sigma_n^2 \begin{bmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1/2 \\ 1/2 & \cdots & 1/2 & 1 \end{bmatrix} \tag{8}$$

where $\sigma_n^2$ is the RDOA noise variance.

When the perfect RDOA measurements $g_{1N}$ are replaced by their noisy counterparts $\tilde{g}_{1N}$, (5) will no longer have a solution except for $N = 3$. Under the Gaussian noise assumption, a maximum likelihood estimate of the solution can be found and is given by [3]

$$\hat{\boldsymbol{s}}_{\mathrm{ML}} = \arg\min_{\boldsymbol{s}} \boldsymbol{h}^T(\boldsymbol{s})\boldsymbol{\Sigma}^{-1}\boldsymbol{h}(\boldsymbol{s}) \tag{9}$$

where

$$\boldsymbol{h}(\boldsymbol{s}) = \begin{bmatrix} \|\boldsymbol{s} - \boldsymbol{r}_2\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| - \tilde{g}_{12} \\ \|\boldsymbol{s} - \boldsymbol{r}_3\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| - \tilde{g}_{13} \\ \vdots \\ \|\boldsymbol{s} - \boldsymbol{r}_N\| - \|\boldsymbol{s} - \boldsymbol{r}_1\| - \tilde{g}_{1N} \end{bmatrix}_{(N-1)\times 1}. \tag{10}$$

Equation (9) does not have a closed-form solution. A numerical solution to (9) can be obtained recursively with the Gauss-Newton (GN) algorithm (also known as the Taylor series method):

$$\begin{aligned} \hat{\boldsymbol{s}}_{\mathrm{ML}}(k) = \hat{\boldsymbol{s}}_{\mathrm{ML}}(k-1) &- (\boldsymbol{J}_{k-1}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{J}_{k-1})^{-1} \\ &\times \boldsymbol{J}_{k-1}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{h}(\hat{\boldsymbol{s}}_{\mathrm{ML}}(k-1)), \quad k = 0, 1, \dots \end{aligned} \tag{11}$$

where $\boldsymbol{J}_k$ is the Jacobian of $\boldsymbol{h}(\boldsymbol{s})$ evaluated at $\boldsymbol{s} = \hat{\boldsymbol{s}}_{\mathrm{ML}}(k)$. A major disadvantage of the GN algorithm is its vulnerability to divergence unless it is initialized sufficiently close to the solution.

## 3. CONSTRAINED WEIGHTED LEAST SQUARES SOLUTION

A closed-form solution to the source location can be derived in several ways (see e.g. [4, 2, 5]). One particularly elegant approach is based on the law of cosines [6]. In the geometry of Fig. 1, firstly consider the triangle formed by the corner points $\boldsymbol{s}$, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Applying the law of cosines to this triangle yields:

$$\|\boldsymbol{d}_2\|^2 = \|\boldsymbol{d}_1\|^2 + \|\boldsymbol{r}_{12}\|^2 - 2\boldsymbol{r}_{12}^T\boldsymbol{d}_1 \tag{12}$$

where

$$\boldsymbol{r}_{ij} = \boldsymbol{r}_j - \boldsymbol{r}_i, \quad i, j \in \{1, \dots, N\}. \tag{13}$$

From (4a), we have

$$\|\boldsymbol{d}_2\|^2 = g_{12}^2 + \|\boldsymbol{d}_1\|^2 + 2g_{12}\|\boldsymbol{d}_1\|. \tag{14}$$

Substituting (14) into (12) yields

$$\begin{bmatrix} \boldsymbol{r}_{12}^T & g_{12} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_1 \\ \|\boldsymbol{d}_1\| \end{bmatrix} = \frac{1}{2}(\|\boldsymbol{r}_{12}\|^2 - g_{12}^2). \tag{15}$$

In a general source localization scenario with $N$ receivers, applying the above steps to each triangle defined by the triplets $\{\boldsymbol{s}, \boldsymbol{r}_1, \boldsymbol{r}_i\}$, $i = 2, \dots, N$, and stacking the resulting row equations (15), we obtain

$$\begin{bmatrix} \boldsymbol{r}_{12}^T & g_{12} \\ \boldsymbol{r}_{13}^T & g_{13} \\ \vdots & \vdots \\ \boldsymbol{r}_{1N}^T & g_{1N} \end{bmatrix}_{(N-1)\times 3} \begin{bmatrix} \boldsymbol{d}_1 \\ \|\boldsymbol{d}_1\| \end{bmatrix}_{3\times 1} = \frac{1}{2} \begin{bmatrix} \|\boldsymbol{r}_{12}\|^2 - g_{12}^2 \\ \|\boldsymbol{r}_{13}\|^2 - g_{13}^2 \\ \vdots \\ \|\boldsymbol{r}_{1N}\|^2 - g_{1N}^2 \end{bmatrix}_{(N-1)\times 1}. \tag{16}$$

After solving the above matrix equation for the unknown range vector $\boldsymbol{d}_1$, the source location is simply given by

$$\boldsymbol{s} = \boldsymbol{r}_1 + \boldsymbol{d}_1. \tag{17}$$

In terms of the noisy RDOA measurements, the matrix equation (16) becomes

$$\boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} + \boldsymbol{\eta} \tag{18}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{r}_{12}^T & \tilde{g}_{12} \\ \boldsymbol{r}_{13}^T & \tilde{g}_{13} \\ \vdots & \vdots \\ \boldsymbol{r}_{1N}^T & \tilde{g}_{1N} \end{bmatrix}, \qquad \boldsymbol{b} = \frac{1}{2} \begin{bmatrix} \|\boldsymbol{r}_{12}\|^2 - \tilde{g}_{12}^2 \\ \|\boldsymbol{r}_{13}\|^2 - \tilde{g}_{13}^2 \\ \vdots \\ \|\boldsymbol{r}_{1N}\|^2 - \tilde{g}_{1N}^2 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{d}_1 \\ \|\boldsymbol{d}_1\| \end{bmatrix}, \qquad \boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{N-1} \end{bmatrix}.$$

Here $\boldsymbol{\eta}$ is the noise vector of the matrix equation (18). Comparison of (16) and (18) yields

$$\eta_i = \frac{1}{2}n_{1,i+1}^2 + n_{1,i+1}\|\boldsymbol{d}_{i+1}\|.$$

The approximate matrix equation $\boldsymbol{Ay} \approx \boldsymbol{b}$ can be solved by using a weighted least squares (WLS) criterion:

$$\hat{\boldsymbol{y}} = \arg\min_{\boldsymbol{y}} (\boldsymbol{Ay} - \boldsymbol{b})^T \boldsymbol{W} (\boldsymbol{Ay} - \boldsymbol{b}) \tag{19}$$

$$= (\boldsymbol{A}^T \boldsymbol{W} \boldsymbol{A})^{-1} \boldsymbol{A}^T \boldsymbol{W} \boldsymbol{b}. \tag{20}$$

The weighting matrix $\boldsymbol{W}$ is given by the inverse covariance matrix of $\boldsymbol{\eta}$. Assuming that $n_{ij}$ is small (i.e., $n_{ij} \approx 0$), the noise vector $\boldsymbol{\eta}$ can be approximated by

$$\boldsymbol{\eta} \approx \begin{bmatrix} n_{12}\|\boldsymbol{d}_2\| \\ n_{13}\|\boldsymbol{d}_3\| \\ \vdots \\ n_{1N}\|\boldsymbol{d}_N\| \end{bmatrix}. \tag{21}$$

Thus, under the small noise assumption, the weighting matrix is

$$\boldsymbol{W} = \boldsymbol{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{T} \tag{22}$$

where

$$\boldsymbol{T} = \text{diag}\{1/\|\boldsymbol{d}_2\|, 1/\|\boldsymbol{d}_3\|, \ldots, 1/\|\boldsymbol{d}_N\|\}.$$

If the source is sufficiently away from the receivers, in which case we have a large range-to-baseline ratio, the diagonal matrix $\boldsymbol{T}$ can be replaced by the identity matrix, leading to

$$\boldsymbol{W} = \boldsymbol{\Sigma}^{-1}. \tag{23}$$

In practice, the large range-to-baseline ratio assumption has to be made unless some prior information is available about the source range.

Given the WLS estimate $\hat{\boldsymbol{y}} = [\hat{y}_1, \hat{y}_2, \hat{y}_3]^T$, the source location estimate is given by

$$\hat{\boldsymbol{s}} = \boldsymbol{r}_1 + [\hat{y}_1, \hat{y}_2]^T. \tag{24}$$

The WLS estimate has two main problems. Firstly, the estimate ignores the dependence between the entries of $\boldsymbol{y} = [y_1, y_2, y_3]^T$, i.e.,

$$y_3^2 = y_1^2 + y_2^2. \tag{25}$$

Secondly, $\hat{\boldsymbol{y}}$ is biased (i.e., $E\{\hat{\boldsymbol{y}}\} \neq \boldsymbol{y}$) because of the correlation between $\boldsymbol{A}$ and $\boldsymbol{\eta}$.

The dependence between the entries of $\boldsymbol{y}$, specified in (25), can be enforced into the solution by means of a nonlinear constraint to improve the estimation performance. To realize the constraint in (25), define a vector function $\boldsymbol{f}(\cdot)$ that transforms a $2 \times 1$ vector to a $3 \times 1$ vector according to

$$\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \boldsymbol{x} \\ \|\boldsymbol{x}\| \end{bmatrix}_{3\times 1}, \quad \boldsymbol{x} \in \mathbb{R}^2. \tag{26}$$

Using the above parameterization, the constrained version of the WLS criterion in (19) becomes

$$\hat{\boldsymbol{d}}_1 = \arg\min_{\boldsymbol{d}_1} (\boldsymbol{A}\boldsymbol{f}(\boldsymbol{d}_1) - \boldsymbol{b})^T \boldsymbol{W} (\boldsymbol{A}\boldsymbol{f}(\boldsymbol{d}_1) - \boldsymbol{b}). \tag{27}$$

The source location estimate is given by $\hat{\boldsymbol{s}} = \boldsymbol{r}_1 + \hat{\boldsymbol{d}}_1$. The constrained WLS criterion does not have a closed-form solution. However, unlike the GN cost function, the CWLS cost function does not lead to divergence, although it may still have multiple local minima [5]. To avoid local minima, we can use the WLS solution as an initialization.

## 4. RECURSIVE SOURCE TRACKING

Suppose that location estimates of a moving source are calculated on a regular basis at $L$ time instants $t_k = kT$, $k = 0, 1, \ldots, L-1$. Let $\hat{\boldsymbol{s}}_k$ denote the CWLS location estimate at time instant $k$. The successive location estimates can be smoothed by using a source motion model incorporating the initial location and velocity, and constant acceleration. The source trajectory can be estimated by utilizing the kinematic equation [7]

$$\boldsymbol{s}_k = \boldsymbol{s}_0 + \boldsymbol{v}_0 t_k + \frac{1}{2}\boldsymbol{a} t_k^2 \tag{28}$$

$$= \boldsymbol{M}_k \boldsymbol{\xi}. \tag{29}$$

Here $\boldsymbol{s}_0$ and $\boldsymbol{v}_0$ are the source location and velocity at $t_0$, respectively, $\boldsymbol{a}$ is the constant source acceleration,

$$\boldsymbol{M}_k = \begin{bmatrix} 1 & 0 & t_k & 0 & \frac{1}{2}t_k^2 & 0 \\ 0 & 1 & 0 & t_k & 0 & \frac{1}{2}t_k^2 \end{bmatrix}$$

and

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{s}_0 \\ \boldsymbol{v}_0 \\ \boldsymbol{a} \end{bmatrix}$$

is the $6 \times 1$ source motion parameter vector. The unknown vector $\boldsymbol{\xi}$ can be estimated from

$$\begin{bmatrix} \boldsymbol{M}_0 \\ \boldsymbol{M}_1 \\ \vdots \\ \boldsymbol{M}_{L-1} \end{bmatrix} \boldsymbol{\xi} \approx \begin{bmatrix} \hat{\boldsymbol{s}}_0 \\ \hat{\boldsymbol{s}}_1 \\ \vdots \\ \hat{\boldsymbol{s}}_{L-1} \end{bmatrix}. \tag{30}$$

In most practical applications, the acceleration can be assumed to be zero (i.e., $\boldsymbol{a} = \boldsymbol{0}$).

To track maneuvering sources, (30) can be solved recursively, using

$$\hat{\boldsymbol{\xi}}_k = \boldsymbol{\Phi}_k^{-1} \boldsymbol{\phi}_k \tag{31}$$

where

$$\boldsymbol{\Phi}_k = \lambda \boldsymbol{\Phi}_{k-1} + \boldsymbol{M}_k^T \boldsymbol{M}_k, \quad k = 0, 1, \ldots \tag{32}$$

$$\boldsymbol{\phi}_k = \lambda \boldsymbol{\phi}_{k-1} + \boldsymbol{M}_k^T \hat{\boldsymbol{s}}_k, \quad k = 0, 1, \ldots \tag{33}$$

with $0 < \lambda < 1$ being the exponential forgetting factor. The recursive algorithm is initialized to $\boldsymbol{\Phi}_{-1} = \boldsymbol{0}$ and $\boldsymbol{\phi}_{-1} = \boldsymbol{0}$. To ensure that $\boldsymbol{\Phi}_k$ is invertible, (31) is not solved until $k \geq 2$. Note that the matrix $\boldsymbol{\Phi}_k^{-1}$ is calculated only once as it is deterministic and independent of location estimates. Therefore, the calculation of $\boldsymbol{\Phi}_k^{-1}$ has no computational complexity. The smoothed location estimates are given by

$$\hat{\hat{\boldsymbol{s}}}_k = \boldsymbol{M}_k \hat{\boldsymbol{\xi}}_k. \tag{34}$$

Setting $\lambda < 1$ allows old source location estimates to be forgotten, which in turn permits tracking of maneuvering sources. By making $\lambda$ small, the tracking performance is improved at the expense of increased variance.

## 5. SIMULATION EXAMPLES

In the simulation examples, we use the source tracking geometry shown in Fig. 2. The receivers are at $\boldsymbol{r}_1 = [0, 0]^T$, $\boldsymbol{r}_2 = [-5, 8]^T$, $\boldsymbol{r}_3 = [4, 6]^T$, $\boldsymbol{r}_4 = [-2, 4]^T$ and $\boldsymbol{r}_5 = [7, 3]^T$. The RDOA noise variance is set to $\sigma_n^2 = 0.002$.

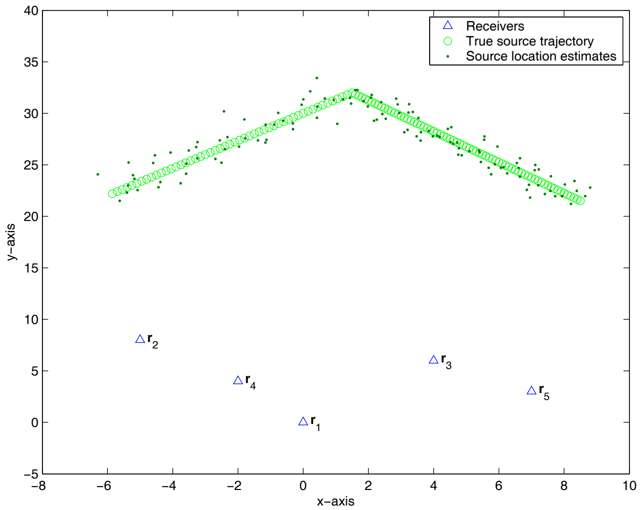


**Fig. 2**. Simulated tracking geometry with location estimates $\hat{s}_k$.

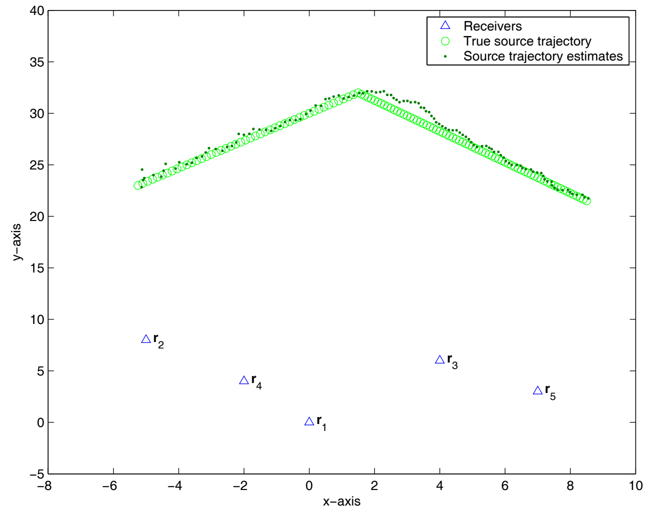


**Fig. 4**. Source trajectory estimates $\hat{\hat{s}}_k$.

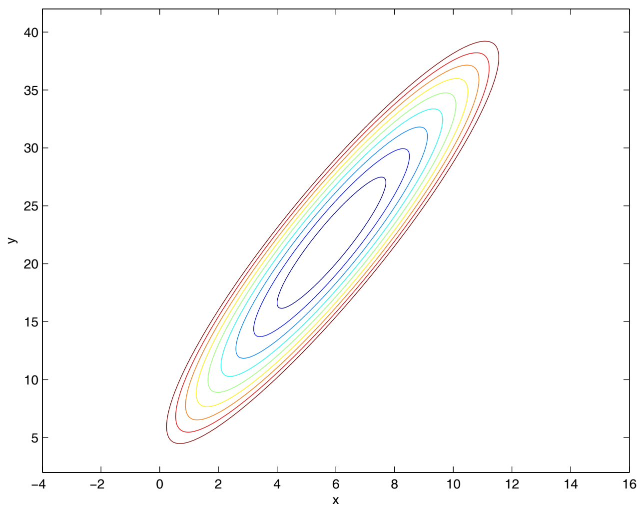


**Fig. 3**. Contour plot of the CWLS cost function.

In the first simulation example, the source is stationary at $s = [6, 22]^T$. The contour plot of the CWLS cost function that is minimized in (27) is shown in Fig. 3. Note how flat the cost function is despite being convex in this case. The bias and MSE of the maximum likelihood GN and CWLS estimates were compared using 10,000 simulation runs. The CWLS algorithm was implemented using the Nelder-Mead simplex method, and initialized to the origin. For the GN algorithm, 10 iterations were used, and the algorithm was initialized to the true solution to avoid divergence. The mean CWLS and GN estimates were found to be $[5.9960, 21.9866]^T$ and $[6.0039, 22.0123]^T$, respectively, and the MSE values were $0.2645$ and $0.2660$. In this case, the CWLS and GN algorithms have an identical performance.

The tracking performance of the recursive algorithm in (34) was simulated. The results for one realization of CWLS location estimates $\hat{s}_k$ are shown in Fig. 4. The CWLS location estimates used in the tracking simulation are shown in Fig. 3 along with the true source trajectory. The TDOA measurements are taken at $t_k = k$, $k = 0, 1, \ldots, L-1$ with $L = 120$. The source moves in two constant-velocity legs. In the first leg, the initial source location and velocity are $s_0 = [-6, 22]^T$ and $v_0 = [0.15, 0.20]^T$, respectively. The second leg starts at $k = 50$ with the velocity changing to $[0.1, -0.15]^T$. The forgetting factor is set to $\lambda = 0.92$. Comparison of Figs. 3 and 4 reveals the improved tracking performance of the recursive estimates $\hat{\hat{s}}_k$ with respect to the stationary CWLS location estimates $\hat{s}_k$.

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