GEOLOCATION BY TIME DIFFERENCE OF ARRIVAL USING HYPERBOLIC ASYMPTOTES

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ABSTRACT

The paper proposes a new simplified algorithm to estimate the location of an emitter by utilizing time difference of arrival (TDOA) measurements. This is achieved by recasting the estimation problem in prolate spheroidal coordinates. Prolate spheroidal coordinates greatly simplify the TDOA equations, producing a set of linear equations in the far field limit. The set of linear equations corresponds to the hyperbolic asymptotes of the TDOA measurements. We also develop a systematic approach that associates the hyperbolic asymptotes with the emitter. In the near field the far-field solution can be used to "seed" the iterative maximum likelihood (ML) estimate, enabling convergence to the ML solution.

1. INTRODUCTION

Geolocation or position location of electromagnetic transmissions has a wide variety of applications. In Australia, geolocation of lightning strikes is used extensively to eliminate small bush fires before they spread over large areas [1]. The requirement for geolocation of mobile phone calls to emergency services (referred to as E-911 in the US) has led to renewed interest in geolocation from the telecommunications industry [2].

Geolocation by time difference of arrival (TDOA) is a computationally attractive approach to passive localization of an emitter. It requires solving a set of nonlinear equations obtained from TDOA measurements. The maximum likelihood (ML) solution to geolocation by TDOA does not have a closed-form solution and requires an iterative gradient-descent algorithm to obtain a numerical solution [3]. Iterative ML solutions exhibit convergence difficulties. Closed-form suboptimal geolocation solutions have been developed to avoid the convergence problems (see e.g. [4, 5, 6]).

In this paper, we simplify the problem of TDOA geolocation by recasting it in prolate spheroidal coordinates. In the far field limit the TDOA equations become linear and hence can be solved analytically as the intersection of hyperbolic asymptotes. A systematic method is proposed to associate the hyperbolic asymptotes with the emitter. In the near field the far field solution can be used to "seed" the iterative ML estimate, enabling convergence to the ML solution.

2. PROBLEM FORMULATION

Geolocation by TDOA, also referred to as range difference of arrival (RDOA), is a technique for determining the position of an emitter from measurements of the time differences of arrival of the emitter's signal at pairs of receivers. Fig. 1 shows a two-dimensional geolocation setup with an array of four receivers at locations $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_4$ and an emitter at $\boldsymbol{r}_{\mathrm{em}}$.

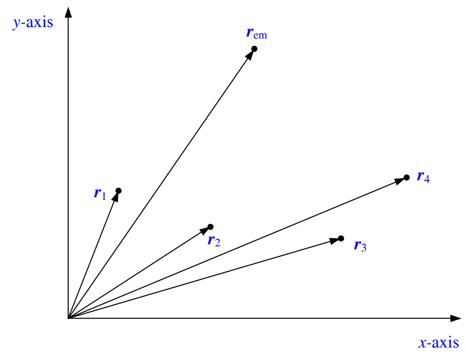

Fig. 1. A typical time difference of arrival setup. Receivers are located at $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_4$ and the emitter at $\boldsymbol{r}_{\mathrm{em}}$.

The time of arrival t_i for a receiver located at $\boldsymbol{r}_i$ of a signal emitted from $\boldsymbol{r}_{\mathrm{em}}$ at time t_{em} is given by

$$t_i = t_{\mathrm{em}} + \frac{\|\boldsymbol{r}_i - \boldsymbol{r}_{\mathrm{em}}\|}{c}, \quad i = 1, \ldots, N \tag{1}$$

where c is the speed of propagation for the emitter signal, N is the number of receivers and $\|.\|$ denotes the Euclidean norm. Equation (1) contains three unknowns $\{t_{\mathrm{em}}, x_{\mathrm{em}}, y_{\mathrm{em}}\}$ and three knowns $\{t_i, x_i, y_i\}$ where $\boldsymbol{r}_{\mathrm{em}} = [x_{\mathrm{em}}, y_{\mathrm{em}}]^T$ and $\boldsymbol{r}_i = [x_i, y_i]^T$ with ^{T} denoting the transpose operator. To solve (1) as a set of nonlinear equations, we would need at least one equation for every unknown parameter, i.e., we need at least three equations, and hence at least three receivers. The first step to solving (1) is determining TDOA between any pair of receivers

$$\Delta t_{ij} \equiv t_i - t_j = \frac{\|\boldsymbol{r}_i - \boldsymbol{r}_{\mathrm{em}}\|}{c} - \frac{\|\boldsymbol{r}_j - \boldsymbol{r}_{\mathrm{em}}\|}{c}, \quad \forall i \neq j \tag{2}$$

thereby eliminating t_{em} and reducing the number of equations by one. There are $N-1$ *linearly independent* TDOA equations for a system consisting of N receivers. For convenience, and without loss of generality, we reference the time of arrival to the Nth receiver so that the $N-1$ linearly independent TDOA equations can be written compactly as

$$\begin{bmatrix} \|\boldsymbol{r}_1 - \boldsymbol{r}_{\text{em}}\| - \|\boldsymbol{r}_N - \boldsymbol{r}_{\text{em}}\| - c\Delta t_{1N} \\ \vdots \\ \|\boldsymbol{r}_{N-1} - \boldsymbol{r}_{\text{em}}\| - \|\boldsymbol{r}_N - \boldsymbol{r}_{\text{em}}\| - c\Delta t_{N-1,N} \end{bmatrix} = \boldsymbol{0}. \quad (3)$$

In practice the Δt_{iN} are not available. Instead we have the noisy TDOA measurements $\Delta\tilde{t}_{iN}$ defined by

$$\Delta\tilde{t}_{iN} = \Delta t_{iN} + n_{iN}$$

where n_{iN}, $i \in \{1, \ldots, N-1\}$, denotes the additive noise corrupting the TDOA measurements, which is usually assumed to be a Gaussian random variable with zero mean and covariance matrix $\boldsymbol{K}$. If the emitter signal received by the sensors is subject to i.i.d. noise, then the covariance matrix becomes [5]

$$\boldsymbol{K} = \sigma_n^2 \begin{bmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1/2 \\ 1/2 & \cdots & 1/2 & 1 \end{bmatrix}$$

where σ_n^2 is the TDOA noise variance.

The task of geolocation by TDOA is to determine the emitter location from the knowledge of the receiver locations and the RDOA measurements using the nonlinear vector equation (3). Equation (3) tells us that we need at least three receivers to provide two linearly independent TDOA equations. From these two equations it is possible to determine the two unknowns of the emitter location $[x_{\text{em}}, y_{\text{em}}]^T$ in the absence of TDOA noise. For a given pair of receivers the solution to (2) is a hyperbola. The intersection of two or more hyperbolae gives the location of the emitter.

3. THE TAYLOR SERIES METHOD

While the emitter location can be seen by the intersection of the hyperbolae pictorially, it is much harder to determine numerically from the nonlinear coupled equations of (3). The reason for this is two-fold. Firstly, in the presence of TDOA noise, (3) will not have a solution, and, secondly, no closed-form solution exists for (3). From now on, we will use the noisy TDOA measurements $\Delta\tilde{t}_{iN}$ in our analysis.

The Taylor-series method [3] effectively linearizes the noisy version of (3) by taking the Taylor series expansion of

$$c\Delta\tilde{t}_{iN} = \|\boldsymbol{r}_i - \boldsymbol{r}_{\text{em}}\| - \|\boldsymbol{r}_N - \boldsymbol{r}_{\text{em}}\| + cn_{iN} \quad (4)$$

about an initial guess of the emitter location $\boldsymbol{r}(0)$, so that (4) becomes

$$\begin{aligned} c\Delta\tilde{t}_{iN} = & \|\boldsymbol{r}_i - \boldsymbol{r}(0)\| - \|\boldsymbol{r}_N - \boldsymbol{r}(0)\| \\ & + \nabla\left(\|\boldsymbol{r}_i - \boldsymbol{r}\| - \|\boldsymbol{r}_N - \boldsymbol{r}\|\right)|_{\boldsymbol{r}=\boldsymbol{r}(0)} (\boldsymbol{r}_{\text{em}} - \boldsymbol{r}(0)) \\ & + cn_{iN} + \mathcal{O}((\boldsymbol{r}_{\text{em}} - \boldsymbol{r}(0))^2). \end{aligned} \quad (5)$$

The gradient row vector is $\nabla = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]$ and $\boldsymbol{r} = [x, y]^T$ is the position column vector. The notation $\mathcal{O}((\boldsymbol{r}_{\text{em}} - \boldsymbol{r}(0))^2)$ refers to second and higher order terms in $\boldsymbol{r}_{em} - \boldsymbol{r}(0)$ which, if $\boldsymbol{r}_{\text{em}}$ and $\boldsymbol{r}(0)$ are sufficiently close, are negligible. By ignoring second and higher order terms and stacking (5) for $i = 1, \ldots, N-1$, we get

$$\boldsymbol{f}_0 + \boldsymbol{J}_0(\boldsymbol{r}_{\text{em}} - \boldsymbol{r}(0)) \approx c\boldsymbol{n}, \quad (6)$$

where $\boldsymbol{f}_0$ is the $(N-1) \times 1$ column vector

$$\boldsymbol{f}_0 = \begin{bmatrix} \|\boldsymbol{r}_N - \boldsymbol{r}(0)\| - \|\boldsymbol{r}_1 - \boldsymbol{r}(0)\| + c\Delta\tilde{t}_{1N} \\ \vdots \\ \|\boldsymbol{r}_N - \boldsymbol{r}(0)\| - \|\boldsymbol{r}_{N-1} - \boldsymbol{r}(0)\| + c\Delta\tilde{t}_{N-1,N} \end{bmatrix}, \quad (7)$$

$\boldsymbol{J}_0$ is the $(N-1) \times 2$ Jacobian matrix of $\boldsymbol{f}_0$

$$\boldsymbol{J}_0 = \begin{bmatrix} \frac{(\boldsymbol{r}_1 - \boldsymbol{r}(0))^T}{\|\boldsymbol{r}_1 - \boldsymbol{r}(0)\|} - \frac{(\boldsymbol{r}_N - \boldsymbol{r}(0))^T}{\|\boldsymbol{r}_N - \boldsymbol{r}(0)\|} \\ \vdots \\ \frac{(\boldsymbol{r}_{N-1} - \boldsymbol{r}(0))^T}{\|\boldsymbol{r}_{N-1} - \boldsymbol{r}(0)\|} - \frac{(\boldsymbol{r}_N - \boldsymbol{r}(0))^T}{\|\boldsymbol{r}_N - \boldsymbol{r}(0)\|} \end{bmatrix} \quad (8)$$

and $\boldsymbol{n} = [n_{1N}, \ldots, n_{N-1,N}]^T$ is the TDOA noise vector.

The Taylor series method uses the weighted least squares estimate of $\boldsymbol{r}_{\text{em}}$ in (6), i.e., $\boldsymbol{r}(0) - (\boldsymbol{J}_0^T \boldsymbol{K}^{-1} \boldsymbol{J}_0)^{-1} \boldsymbol{J}_0^T \boldsymbol{K}^{-1} \boldsymbol{f}_0$, to update the estimated position of the emitter using the recursion

$$\boldsymbol{r}(k+1) = \boldsymbol{r}(k) - (\boldsymbol{J}_k^T \boldsymbol{K}^{-1} \boldsymbol{J}_k)^{-1} \boldsymbol{J}_k^T \boldsymbol{K}^{-1} \boldsymbol{f}_k, \quad k = 0, 1, \ldots \quad (9)$$

The emitter location estimate occurs at the stationary point $\boldsymbol{r}(k+1) = \boldsymbol{r}(k)$. However, in practice, this will only occur when $k \to \infty$. Therefore, for practical reasons, the iterations of (9) are stopped when $\|\boldsymbol{r}(k+1) - \boldsymbol{r}(k)\| < \epsilon$ where ϵ is some predefined threshold value.

The iterative Taylor series method derived above can be shown to be equivalent to the Gauss-Newton implementation of the ML estimate of the emitter location [3]. The ML cost function minimized by the Taylor series method is

$$J_{\text{ML}}(\boldsymbol{r}) = \boldsymbol{f}^T(\boldsymbol{r}) \boldsymbol{K}^{-1} \boldsymbol{f}(\boldsymbol{r}) \quad (10)$$

where $\boldsymbol{f}(\boldsymbol{r}) = \boldsymbol{f}_k|_{\boldsymbol{r}(k)=\boldsymbol{r}}$. The Taylor series method can provide accurate and robust results. However, it requires a good initial guess $\boldsymbol{r}(0)$; otherwise, (9) may diverge because of the nonconvexity of the ML cost function. The requirement of an accurate initial guess is the major weakness of the Taylor series method. We are going to propose a remedy for this problem in the following sections.

4. PROLATE SPHEROIDAL COORDINATES AND TDOA

The TDOA equations as expressed in (2) simplify greatly if they are written in prolate spheroidal coordinates. In this section we will recast (2) in prolate spheroidal coordinates and show how from the resulting equation two lines of bearing can be obtained.

In three-dimensional space, prolate spheroidal coordinates are a set of curvilinear coordinates which are related to Cartesian coordinates $\{x, y, z\}$ by [7]:

$$\begin{aligned} x &= a\cosh\xi\cos\eta \\ y &= a\sinh\xi\sin\eta\sin\phi \\ z &= a\sinh\xi\sin\eta\cos\phi \end{aligned} \quad (11)$$

where a is a constant scaling parameter.

Let us write the TDOA equation (2) in prolate spheroidal coordinates. We start by considering a Cartesian coordinate system $\{x', y', z'\}$ in which two receivers are located at $x' = \pm a$ and $y' = z' = 0$, in which case

$$\begin{aligned} c\Delta t &= \sqrt{(x'+a)^2 + y'^2 + z'^2} - \sqrt{(x'-a)^2 + y'^2 + z'^2} \\ &= 2a\cos\eta. \end{aligned}$$

The sign ambiguity of η in $\cos\eta$ means that for a given Δt there are two possible values of η,

$$\eta = \pm \arccos\left(\frac{c\Delta t}{2a}\right). \tag{12}$$

For two-dimensional geolocation problems, we have $\phi = \pi/2$ in prolate spheroidal coordinates, so that

$$\begin{aligned} x' &= a\cosh\xi\cos\eta. \\ y' &= a\sinh\xi\sin\eta \end{aligned} \tag{13}$$

Not only does the fact that η is a constant for a given Δt enable us to quickly determine the TDOA hyperbola, but it also gives us the emitter bearing in the far field limit. To see this, consider the limit $\xi \to \infty$ of (13):

$$\begin{aligned} \lim_{\xi\to\infty} x' &= ae^{\xi}\cos\eta \\ \lim_{\xi\to\infty} y' &= ae^{\xi}\sin\eta \end{aligned}$$

which means that

$$\begin{aligned} \lim_{\xi\to\infty} \frac{y'}{x'} &= \tan\eta \\ &= \tan\left(\pm\arccos\left(\frac{\Delta t}{2a}\right)\right). \end{aligned}$$

Hence the bearing can be easily related to the time difference of arrival Δt and the separation $2a = \|\boldsymbol{r}_j - \boldsymbol{r}_i\|$ between the receivers. The bearing angle from the mid-point of a pair of receivers located at $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ is

$$\theta = \alpha + \eta. \tag{14}$$

The angle α is the bearing of the x' axis with respect to the x axis, hence

$$\alpha = \mathrm{arcTan}\left(\frac{y_j - y_i}{x_j - x_i}\right) \tag{15}$$

where arcTan[1] is the same as arctan except that it includes information about quadrant of the argument, e.g., $\mathrm{arcTan}(\frac{-1}{-1}) = \frac{-3\pi}{4}$ whereas $\arctan(\frac{-1}{-1}) = \frac{\pi}{4}$. Note that the angles α, η and θ are bearings so that α and θ are zero on the x axis and $\eta = 0$ defines the x' axis; angles are positive in the anti-clockwise direction.

Equations (12), (14) and (15) enable us to determine the bearing angles $\theta_{ij}^{\pm}$ of the hyperbolic asymptotes with respect to the horizontal axis associated with a pair of receivers located at $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ whose TDOA measurement is $\Delta\tilde{t}_{ij}$:

$$\boxed{\theta_{ij}^{\pm} = \mathrm{arcTan}\left(\frac{y_j - y_i}{x_j - x_i}\right) \pm \arccos\left(\frac{c\Delta\tilde{t}_{ij}}{\|\boldsymbol{r}_j - \boldsymbol{r}_i\|}\right).} \tag{16}$$

5. EMITTER LOCALIZATION USING HYPERBOLIC ASYMPTOTES

Each TDOA hyperbola produces a pair of asymptotes with bearing angles defined in (16). For N receivers, there will be $2(N-1)$ hyperbolic asymptotes, which we will refer to as bearing lines. Out of $2(N-1)$ bearing lines, only $N-1$ of them are associated with the emitter and, therefore, can be used for localization purposes.

[1] The Matlab equivalent of arcTan is atan2.

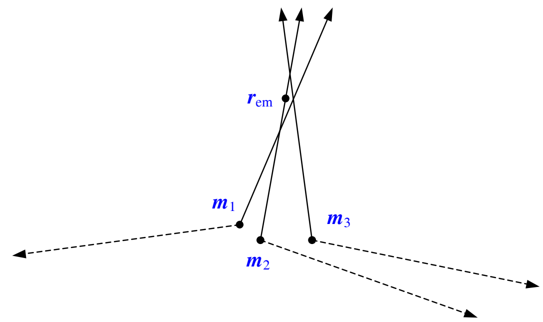

Fig. 2. Bearing lines for a geolocation scenario with $N = 4$ receivers.

Fig. 2 shows an example for $N = 4$ receivers. The bearing lines emanate from midpoints of receiver pairs defined by

$$\boldsymbol{m}_i = \frac{1}{2}(\boldsymbol{r}_i + \boldsymbol{r}_N). \tag{17}$$

The association of bearing lines with the emitter does not always have a straightforward rule-based solution, except possibly for the case of fixed receivers. If the receivers are moving, finding the associated bearing lines becomes trickier because the geometry changes continuously.

To associate bearings lines the first thing we do is a feasibility check. This is done by determining if a given bearing line intersects with another bearing line from a different midpoint. If no intersection occurs that bearing is removed from our list of bearings. This process is repeated for all bearings and at the end a final list of feasible bearings and midpoints is created.

To triangulate the feasible bearings we use the pseudolinear estimator [8] based on least squares to estimate the emitter location from the feasible bearing lines. The pseudolinear estimate of $\boldsymbol{r}_{\mathrm{em}}$ is

$$\hat{\boldsymbol{r}}_{\mathrm{LS}} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b} \tag{18}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \sin\varphi_1 & -\cos\varphi_1 \\ \vdots & \vdots \\ \sin\varphi_{N-1} & -\cos\varphi_{N-1} \end{bmatrix}_{(N-1)\times 2}$$

$$\boldsymbol{b} = \begin{bmatrix} [\sin\varphi_1, -\cos\varphi_1]\boldsymbol{m}_1 \\ \vdots \\ [\sin\varphi_{N-1}, -\cos\varphi_{N-1}]\boldsymbol{m}_{N-1} \end{bmatrix}_{(N-1)\times 1}.$$

Here $\{\boldsymbol{m}_1, \varphi_1\}, \ldots, \{\boldsymbol{m}_{N-1}, \varphi_{N-1}\}$ is the list of feasible bearing lines with $\varphi_i \in \{\theta_{iN}^{+}, \theta_{iN}^{-}\}$. The resulting pseudolinear estimate provides an excellent initial guess for the Taylor series method. A weighted version of the pseudolinear estimate can also be considered to improve the estimation accuracy of (18) [9].

6. SIMULATIONS

The simulated geolocation setup and TDOA hyperbolae resulting from noisy TDOA measurements are shown in Fig. 3. The true emitter location is $\boldsymbol{r}_{\mathrm{em}} = [4, 30]^T$. The receivers are located at $\boldsymbol{r}_1 = [5, 3]^T$, $\boldsymbol{r}_2 = [2, 0]^T$, $\boldsymbol{r}_3 = [-3, 2]^T$ and $\boldsymbol{r}_4 = [0, 0]^T$.

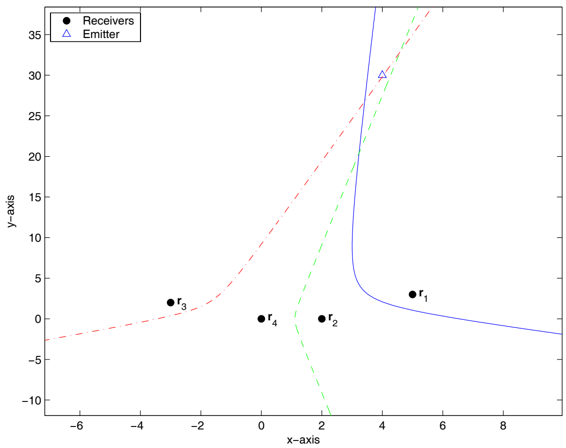

Fig. 3. Simulated TDOA geolocation setup.

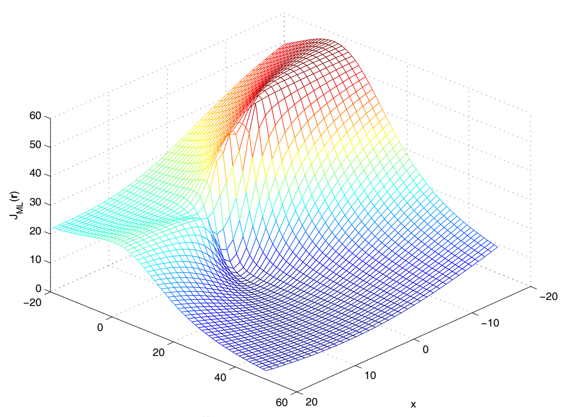

Fig. 4. Maximum likelihood cost function.

The TDOA noise variance is $\sigma_n^2 = 0.004/c^2$. The ML cost function for the simulated scenario of Fig. 3 is depicted in Fig. 4. Note how irregular the cost function topology is. It is not only non-convex, but also it appears to have a downward slope behind the receivers on the opposite side of the emitter. This is the reason for divergence experienced by gradient-descent algorithms when initialized poorly. These particular features of the ML cost function topology reinforce the need for good initialization. In Fig. 5 we illustrate the application of emitter localization based on hyperbolic asymptotes. There are six bearing lines, three of which are feasible. For completeness we have included all eight bearing line combinations and plotted the pseudolinear estimates for them in Fig. 5. The feasible bearing lines yielded the estimate shown just below the emitter in Fig. 5. This estimate is used as the initial guess for the Taylor series method. In this case only two iterations of the Taylor series method gives the ML estimate of the emitter location.

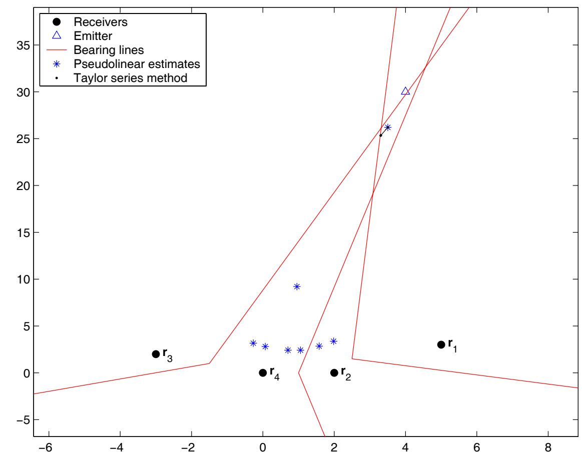

Fig. 5. Geolocation using the Taylor series method with initial guess obtained from the pseudolinear estimator.

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