EFFICIENT DOA ESTIMATION METHOD EMPLOYING UNITARY IMPROVED POLYNOMIAL ROOTING

Erwei Lin†, *Li Bai*‡ and *Moshe Kam*†

† Data Fusion Laboratory, Dept. of ECE
Drexel University
Philadelphia, PA, 19104, USA
Phone: (215) 895-6921
{erwei, kam}@minerva.ece.drexel.edu

‡ Computer Fusion Laboratory, Dept. of ECE
Temple University
Philadelphia, PA, 19122, USA
Phone: (215) 204-6616
lbai@temple.edu

ABSTRACT

High resolution direction-of-arrival (DOA) estimation is one of the most challenging problems in array signal processing. In this paper, a variation of the Improved Polynomial Rooting (IPR) method is proposed for DOA estimation of multiple targets by a sensor array. The variation, Unitary IPR (UIPR), transforms the complex-valued covariance matrix of the sensor signals to a real-valued matrix using unitary transformations. Then the IPR method is applied to determine the DOA of the targets. Simulation results indicate the potential improvement provided by our approach compared with MUSIC, Root-MUSIC, ESPRIT, and IPR.

1. INTRODUCTION

Over the last 30 years, direction of arrival (DOA) estimators for sensor arrays have received considerable attention [7, 11]. These techniques are widely used in applications in sonar, radar, radio, telecommunication, astronomy and strategic defense operations. Recently, renewed interest was expressed in DOA for wireless communication applications (*e.g,* smart antenna and mobile unit location detection [6]).

The general objective is to use sensor arrays to estimate the number of distinct signals in the array's volume of surveillance, the directions of arrival (DOA) of these signals, and their carrier frequencies. There are several high resolution methods such as MUSIC, Root-MUSIC, MIN-NORM, ESPRIT and the IPR [1–3] designed for this purpose. These methods often require singular value decomposition (SVD) in the complex-valued signal subspace. The resulting computations are often expensive, especially during eigen-decomposition. Other methods, based on maximum likelihood principles, have narrower scope of applicability because they depend on accurate target signal distribution function. To reduce the computational complexity during eigen-decomposition, the unitary Root MUSIC [9] and unitary ESPRIT [8] methods were proposed for DOA estimation, using real-valued SVD.

In this paper, a variant of the IPR method is developed, taking advantage of unitary transformations. In general, the IPR method improves DOA estimation by using reduced order characteristic polynomial rooting though Gaussian column reduction. The Unitary IPR method reduces the computational complexity of IPR by using real valued SVD, while maintaining the original precision. The rest of the paper is organized as follows: section 2 discusses the array signal model. Section 3 describes the IPR method. Unitary-IPR method is proposed in section 4. Computer simulations are presented in section 5, comparing UIPR with other methods.

2. PROBLEM FORMULATION

Consider narrow-band signals emitted from q targets from the far field. The signals are assumed to be wide sense stationary processes. They are impinging on an uniform linear array (ULA) of p sensors with inter-sensor distance d.

The q target signals have a known carrier frequency ω_0. The i^{th} signal is emitted in (azimuth) directions θ_i, where $1 \le i \le q$ and $-\frac{\pi}{2} \le \theta_i \le \frac{\pi}{2}$. The output of the p sensors ($p > q$), is modeled as the sum of the signals received from the targets imbedded in additive white Gaussian noise. Let $\boldsymbol{Y}(t) \in C^p$ represent the output of the array where C is the set of complex numbers. We model the array's output as

$$\boldsymbol{Y}(t) \quad = \quad \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{S}(t) + \boldsymbol{\eta}(t), \qquad (1)$$

where $\boldsymbol{S}(t) \in C^q$ is a vector of the q target signals, $\boldsymbol{\eta}(t) \in C^p$ is a white Gaussian noise vector and $\boldsymbol{A}(\boldsymbol{\theta}) = [\boldsymbol{a}(\theta_1), ..., \boldsymbol{a}(\theta_q)]$ is the $p \times q$ matrix of steering vectors. Specifically,

$$\boldsymbol{a}(\theta_i) = [1 \quad e^{-j(\frac{w_0 d}{c})\sin\theta_i} \cdots e^{-j(\frac{w_0 (p-1) d}{c})\sin\theta_i}]^T, \qquad (2)$$

where d is the distance between two adjacent sensors, c is the speed of wave propagation, and $j = \sqrt{-1}$.

Assume that the signal $\boldsymbol{S}$(t) has zero mean and the covariance matrix $\boldsymbol{\Gamma}_s = E(\boldsymbol{S}(t)\boldsymbol{S}^\dagger(t))$, where $\boldsymbol{S}^\dagger(t)$ is the Hermitian conjugate of $\boldsymbol{S}(t)$. The noise covariance matrix is $\boldsymbol{\Gamma}_n = E(\boldsymbol{\eta}(t)\boldsymbol{\eta}^\dagger(t)) = \sigma^2\boldsymbol{I}_p$, where $\boldsymbol{I}_p$ is $p \times p$ identity matrix. If the signals are statistically independent of each other, as well as of the additive noise, the spatial correlation matrix of $\boldsymbol{Y}(t)$ becomes,

$$\boldsymbol{\Sigma} = E(\boldsymbol{Y}(t)\boldsymbol{Y}^\dagger(t)) = \boldsymbol{A}\boldsymbol{\Gamma}_s\boldsymbol{A}^\dagger + \sigma^2\boldsymbol{I}_p. \tag{3}$$

We estimate the correlation matrix, $\boldsymbol{\Sigma}$, by calculating

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{N}\sum_{t=1}^{N}\boldsymbol{Y}(t)\boldsymbol{Y}^\dagger(t), \tag{4}$$

where N is the number of samples.

There exists [4] an orthogonal eigenvector matrix $\boldsymbol{E}$ and a real diagonal eigenvalue matrix $\boldsymbol{\Lambda}$ such that

$$\boldsymbol{\Sigma} = \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}^\dagger, \tag{5}$$

where $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \cdots, \lambda_p)$, $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_p$, and the matrix of eigenvectors is $\boldsymbol{E} = [\boldsymbol{e}_1 \cdots \boldsymbol{e}_q \vdots \boldsymbol{e}_{q+1} \cdots \boldsymbol{e}_p]$.

3. IMPROVED POLYNOMIAL ROOTING

Improved Polynomial Rooting (IPR) [1, 3] is an efficient method to estimate DOA. Let $\boldsymbol{E}_n = [\boldsymbol{e}_{q+1}, \boldsymbol{e}_{q+2}, \cdots \boldsymbol{e}_p]$ be the matrix of noise eigenvectors, corresponding to the $p-q$ smallest eigenvalues of $\boldsymbol{\Lambda}$. Through Gaussian column reduction or Householder transformation, IPR reduces $\boldsymbol{E}_n$ to the form

$$\boldsymbol{G} = \begin{pmatrix} g_{1,q+1} & 0 & \cdots & 0 \\ g_{2,q+1} & g_{2,q+2} & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ g_{q+1,q+1} & g_{q+1,q+2} & \ddots & \hat{g}_{q+1,p} \\ 0 & \hat{g}_{q+2,q+2} & \ddots & g_{q+2,p} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & g_{p,p} \end{pmatrix}_{p\times(p-q)} . \tag{6}$$

The reduced polynomial rooting equation is given as,

$$g(z) = \sum_{i=1}^{q+1} g_i z^{i-1} = 0, \tag{7}$$

where $g_i = \frac{1}{p-q}(g_{i,q+1} + g_{i+1,q+2} + \cdots + g_{i+p-q-1,p})$. One can solve for exactly q roots z_i from equation (7), and the DOA θ_i is obtained from

$$\theta_i = \arcsin\left[Real\left(-\frac{c\ln(z_k)}{j\omega_0 d}\right)\right], \tag{8}$$

where $1 \leq i \leq q$.

4. THE PROPOSED METHOD– UNITARY-IPR

A matrix $\boldsymbol{\Sigma} \in C^{p\times p}$ is called centro-Hermitian matrix [5] if there exists a matrix $\boldsymbol{J}_p$ such that

$$\boldsymbol{\Sigma} = \boldsymbol{J}_p\boldsymbol{\Sigma}^*\boldsymbol{J}_p, \tag{9}$$

where $\boldsymbol{\Sigma}^*$ represents the conjugate of $\boldsymbol{\Sigma}$, and $\boldsymbol{J}_p$ is the exchange matrix,

$$\boldsymbol{J}_p = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & 1 & \cdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}_{p\times p} .$$

A forward-backward (FB) matrix [8] $\widehat{\boldsymbol{\Sigma}}_{FB} \in C^{p\times p}$ can be derived from the estimated signal covariance matrix $\widehat{\boldsymbol{\Sigma}}$ as

$$\widehat{\boldsymbol{\Sigma}}_{FB} = \frac{1}{2}(\widehat{\boldsymbol{\Sigma}} + \boldsymbol{J}_p\widehat{\boldsymbol{\Sigma}}^*\boldsymbol{J}_p) = \boldsymbol{A}\tilde{\boldsymbol{\Gamma}}\boldsymbol{A}^\dagger + \sigma^2\boldsymbol{I}_p, \tag{10}$$

where $\tilde{\boldsymbol{\Gamma}} = \frac{1}{2}(\boldsymbol{\Gamma} + \boldsymbol{D}\boldsymbol{\Gamma}^*\boldsymbol{D}^\dagger)$, and
$\boldsymbol{D} = diag\{e^{-j(\frac{w_0 d(p-1)}{c})\sin\theta_1}, \ldots, e^{-j(\frac{w_0 d(p-1)}{c})\sin\theta_q}\}$.

$\widehat{\boldsymbol{\Sigma}}_{FB}$ is a centro-Hermitian matrix. Like any centro-Hermitian matrices, $\widehat{\boldsymbol{\Sigma}}_{FB}$ can be transformed to a real-valued (unitary) matrix $\boldsymbol{L} \in R^{p\times p}$ [8] using

$$\boldsymbol{L} = \boldsymbol{K}^\dagger\widehat{\boldsymbol{\Sigma}}_{FB}\boldsymbol{K}, \tag{11}$$

where $\boldsymbol{K}$ is a matrix defined as:

$$\boldsymbol{K} = \begin{cases} \frac{1}{\sqrt{2}}\begin{pmatrix} I_{\frac{p}{2}} & jI_{\frac{p}{2}} \\ J_{\frac{p}{2}} & -jJ_{\frac{p}{2}} \end{pmatrix} & \text{if } p \text{ is even number} \\ \frac{1}{\sqrt{2}}\begin{pmatrix} I_{\frac{p-1}{2}} & 0 & jI_{\frac{p-1}{2}} \\ 0 & \sqrt{2} & 0 \\ J_{\frac{p-1}{2}} & 0 & -jJ_{\frac{p-1}{2}} \end{pmatrix} & \text{if } p \text{ is odd number.} \end{cases}$$

To obtain the real-valued matrix $\boldsymbol{L}$ directly via the complex valued covariance matrix $\widehat{\boldsymbol{\Sigma}}$, we have

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{K}^\dagger\widehat{\boldsymbol{\Sigma}}_{FB}\boldsymbol{K} \\ \boldsymbol{L} &= \frac{1}{2}(\boldsymbol{K}^\dagger\widehat{\boldsymbol{\Sigma}}\boldsymbol{K} + (\boldsymbol{K}^*)^\dagger\widehat{\boldsymbol{\Sigma}}^*\boldsymbol{K}^*) \\ \boldsymbol{L} &= Real\{\boldsymbol{K}^\dagger\widehat{\boldsymbol{\Sigma}}\boldsymbol{K}\}. \end{aligned} \tag{12}$$

$\boldsymbol{L}$ can be obtained from either equation (11) or equation (12). However, equation (12) is simpler because there is no need to calculate $\widehat{\boldsymbol{\Sigma}}_{FB}$ explicitly.

Consider the real-valued eigen-decomposition of L

$$\boldsymbol{L} = \boldsymbol{V}\boldsymbol{\Pi}\boldsymbol{V}^\dagger, \tag{13}$$

where $\boldsymbol{\Pi} = \text{diag}(\pi_1, \cdots, \pi_p)$; $\pi_i, i = 1, \cdots, p$ are the eigenvalues of $\boldsymbol{L}$; and $\pi_1 \geq \pi_2 \geq \cdots \pi_p$. Let $\boldsymbol{V} = [v_1 \cdots$

$v_q \vdots v_{q+1} \cdots v_p]$ be corresponding the eigenvector matrix, and $\boldsymbol{V}_n = [v_{q+1}, v_{q+2}, \cdots v_p]$.

$\widehat{\boldsymbol{\Sigma}}_{FB}$ and $\boldsymbol{L}$ are similar matrices in Eq.(11), and the SVD of $\widehat{\boldsymbol{\Sigma}}_{FB}$ is,

$$\widehat{\boldsymbol{\Sigma}}_{FB} = \boldsymbol{U}\boldsymbol{\Pi}\boldsymbol{U}^{\dagger} \tag{14}$$

Equation (14) can be rewritten as $\boldsymbol{\Sigma}_{FB}\boldsymbol{U} = \boldsymbol{U}\boldsymbol{\Pi}$, and we have,

$$\begin{aligned} \boldsymbol{K}^{\dagger}\widehat{\boldsymbol{\Sigma}}_{FB}\boldsymbol{K}\boldsymbol{K}^{\dagger}\boldsymbol{U} &= \boldsymbol{K}^{\dagger}\widehat{\boldsymbol{\Sigma}}_{FB}\boldsymbol{U} \\ \boldsymbol{L}\boldsymbol{K}^{\dagger}\boldsymbol{U} &= \boldsymbol{K}^{\dagger}\boldsymbol{U}\boldsymbol{\Pi} \\ \boldsymbol{L} &= \boldsymbol{K}^{\dagger}\boldsymbol{U}\boldsymbol{\Pi}\boldsymbol{K}\boldsymbol{U}^{\dagger} \end{aligned} \tag{15}$$

From equations (13) and (15), we have

$$\boldsymbol{V} = \boldsymbol{K}^{\dagger}\boldsymbol{U}, \tag{16}$$

or

$$\boldsymbol{U} = \boldsymbol{K}\boldsymbol{V}. \tag{17}$$

Hence,

$$\boldsymbol{U}_n = \boldsymbol{K}\boldsymbol{V}_n. \tag{18}$$

Once $\boldsymbol{U}_n$ is constructed, the IPR method can be used to determine the DOA as described in section 3. The process of the UIPR is summarized as follows.

1	Determine covariance matrixes $\widehat{\boldsymbol{\Sigma}}$ from the sensors' observations
2	Obtain a real-value matrix $\boldsymbol{L}$ from Eq.(12)
3	Solve $\boldsymbol{\Pi}$ and $\boldsymbol{V}$ from $\boldsymbol{L}$
4	Obtain $\boldsymbol{V}_n = [\boldsymbol{v}_1 \cdots \boldsymbol{v}_{q+1}]$ from $\boldsymbol{V}$
5	Calculate $\boldsymbol{U}_n$ using Eq.(18)
6	Column reduce $\boldsymbol{U}_n$ into $\boldsymbol{G}$ (Eq.(6))
7	Sum the values in each column of $\boldsymbol{G}$ to obtain the value of g_i
8	Solve the roots z_k from the polynomial equation (Eq.(7))with coefficients g_i, where $1 \le i \le q+1$ and $1 \le k \le q$
9	Determine the DOAs by $\theta_k = \arcsin(Real(-\frac{c\ln(z_k)}{j\omega_0 d}))$

The floating point operation cost for evaluating the eigenvectors and eigenvalues of $\widehat{\boldsymbol{\Sigma}}_{FB}$ directly is $4p^2$ multiplications and $4p^2$ additions. If we use the real-value eigen-decomposition (13), we need only p^2 multiplications and p^2 additions. We thus gain 75% computational cost efficiency.

5. SIMULATIONS

In order to demonstrate the potential of the UIPR method, we simulated the DOA capability of a sensor array with UIPR and compared with MUSIC, Root-MUSIC, TLS-ESPRIT and IPR.

5.1. Comparisons of RMSE and FPOC vs SNR

Assume we have three uncorrelated targets of equal power located at $\theta_1 = -20^o$, $\theta_2 = 10^o$ and $\theta_3 = 40^o$. Let $d = 10m$ (approximately equal to a half of the wavelength), $p = 10$ and $\omega_0 = 10^8 rad/s$. We ran a simulation for DOA calculation for 1000 times with the number of samples $N = 200$, and examined the accuracy and computational complexity of DOA estimation. The accuracy is determined by Root Mean Square Error (RMSE), and computational complexity is measured by number of Floating Point Operation Counts (FPOCs).

The RMSE is plotted vs. SNR (from $-5dB$ to $15dB$) in Fig. 1. The UIPR estimator shows improvement compared to IPR and TLS-ESPRIT (e.g. 0.14 degree of the error with UIPR at SNR=0 dB compared with 0.16 degree of error with TLS-ESPRIT). It is close to IPR, but achieves this performance with much fewer calculation (Fig. 2). In fact it is computationally superior to all alternatives

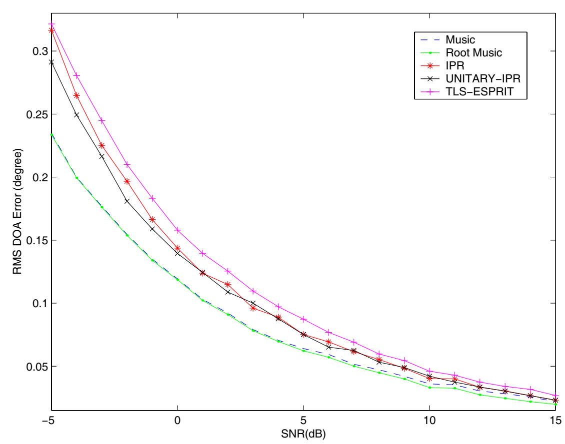

Fig. 1. The RMSE vs. SNR performance of UIPR compared to MUSIC, root-MUSIC, TLS-ESPRIT, and IPR($q = 3$, $p = 10$, and 200 snapshots)

5.2. Comparisons of RMSE and FPOC vs Number of Sensors

We kept SNR=5dB and varied the number of sensors. We show RMSE (Fig. 3) and FPOC (Fig. 4). The number of sensors varies from 5 to 25. Again, UIPR shows a significant computational improvement over other methods while maintaining the error of direction estimation similar to IPR. As expected, the FPOC increase as the number of number of sensors increases.

6. CONCLUSION

We present a computationally efficent version of IPR for DOA estimation, using unitary transformations. The poten-

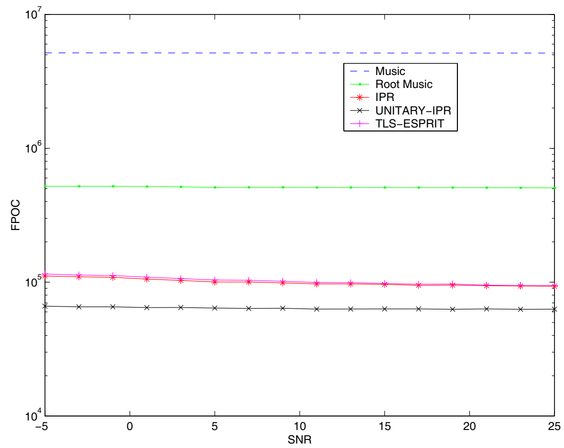

Fig. 2. The Floating Point Operation Counts vs. SNR performance of Unitary-IPR compared to other methods($q = 3$, $p = 10$, and 200 snapshots)

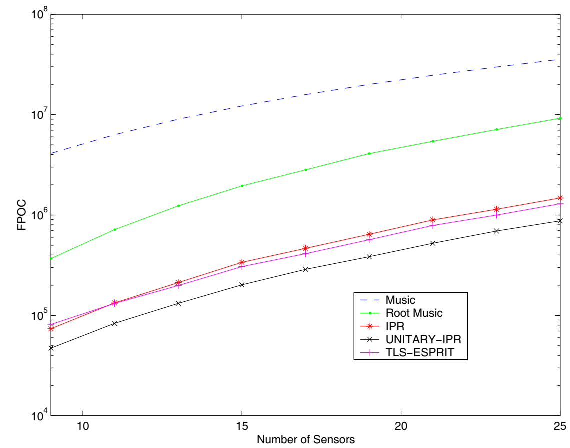

Fig. 4. Comparison of FPOC vs. Number of Sensors(q=3, SNR=5dB, 200 snapshots)

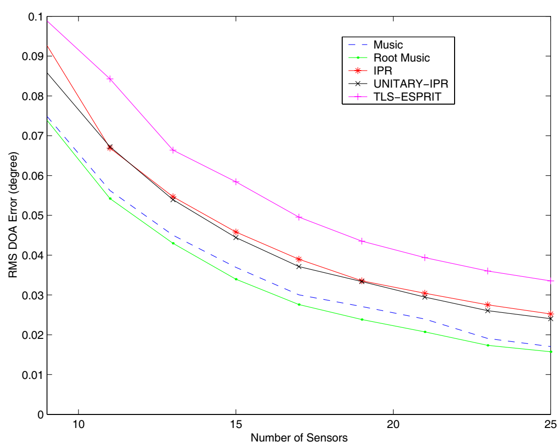

Fig. 3. Comparison of RMSE vs. Number of Sensors (q=3, SNR=5dB, 200 snapshots)

tial of the proposed algorithm is examined through computer simulations.

It appears the UIPR method may be able to estimate the DOA with less computational complexity than popular competing methods while maintaining comparable direction accuracy.

7. REFERENCES

[1] Z.D. Bai, P.R. Krishnaih, and L.C. Zhao, "On the Direction of Arrival Estimation," *Tech. Report*, Center for Multivariate Analysis, University of Pittsburgh. No. 87-12, 1987

[2] Z.D. Bai, P.R. Krishnaih, and L.C. Zhao, "On Rates of Convergence of Efficient Detection Criteria in Signal Processing with White Noise," *IEEE Trans. Info. Theory* vol. 35 no. 2, 380-388, 1989

[3] C. Peng, Directions of arrival estimation and association detection in unknown noise fields, *Ph.D thesis, Dept. of Statistics*, Temple University, Philadelphia, 1999.

[4] S. Haykin, J.P. Reily, V. Kezys, and E. Vertatschitsch, "Some Aspects of Array Signal Processing," *Spatial Statistics and Digital Image Analysis,* National Academy Press, 1992

[5] R. Hill, R. Bates, and S. Waters, "On Centro-Hermitian Matrices," *Siam J. Matrix Anl. Appl.,* vol. 11, pp. 128-133, Jan. 1990

[6] S. Jeng, C. Huang, and T. Huang, "Performance Evaluation of DOA Based Beamforming in W-CDMA System," *IEEE Vehicular Technology Conference,* vol. 7-11, 2652-2656, 2001

[7] R. Kumaresan, and D.W. Tufts, "Estimating the Angles of Arrival of Multiple Plane Waves," *IEEE. Trans. Aero. and Elec.,* vol. 19, no. 1, 134-138, 1983

[8] D.A. Linebarger, R.D. Degroat, and E.M. Dowling, "Efficient direction-finding methods employing forward-backward averaging," *IEEE Trans. Signal Processing,* vol. 42, pp. 2136-2145, Aug. 1994

[9] M. Pesavento, A. Gershman and M. Haardt, "Unitary Root-MUSIC with a Real-Valued Eigendecomposition: A Theoretical and Experimental Performance Study," *IEEE Trans. Signal Processing* vol. 48, No 5, pp1306-1314, May 2000

[10] S. Pillai and B. Kwon, "Forward-backward spatial smoothing techniques for the coherent signal identification," *IEEE Trans. Acoust., Speech, Signal Processing,* vol. 37, pp8-15, Jan. 1989

[11] R.O. Schmidt, Multiple Emitter Location and Signal Parameter Estimation, *Proc. RADC Spectral Estimation Workshop,* 243-258, 1979