STOCHASTIC CRAMER-RAO BOUND OF DOA ESTIMATES FOR NON-CIRCULAR GAUSSIAN SIGNALS

Habti Abeida, Jean-Pierre Delmas

GET/INT, Département CITI, UMR-CNRS 5157, Evry, France

ABSTRACT

This paper focuses on the stochastic Cramer-Rao bound (CRB) on direction of arrival (DOA) estimation accuracy for non-circular Gaussian sources. We derive an explicit expression of the CRB for DOA parameters alone in the case of non-circular complex Gaussian sources by two different methods. One of them consists in computing the asymptotic covariance matrix of the maximum likelihood (ML) estimator, and the other is obtained directly from an extended Slepian-Bangs formula. Finally some properties of this CRB are proved.

1. INTRODUCTION

Deterministic and stochastic CRB's play an important role in array processing because the statistical performances of numerous estimation methods are known to be comparable to these bounds under certain mild conditions. Although the deterministic CRB is known to be not achieved, the stochastic CRB can be achieved asymptotically (in the number of measurements) by the stochastic ML method. But all the contributions on the stochastic CRB are dedicated to circular Gaussian distributions for which an explicit formula was first derived in an indirect form using the asymptotic covariance matrix of the ML estimator [1], [2]. Then this formula has been directly proved from the complex circular Slepian-Bangs formula in [4].

The importance of this circular complex Gaussian CRB formula lies in the fact that under rather general conditions, the circular complex Gaussian CRB matrix is the largest CRB matrix, among the class of arbitrary circular complex distributions with given mean and covariance matrices (see e.g., [3, p. 293]). However non-circular complex signals are frequently encountered in digital communications. For example, binary phase shift keying and offset quaternary phase shift keying are often used. But no closed-form expression of the CRB is available for these signals. Consequently for such non-circular complex signals, we need an upper bound of this CRB.

In this paper, we derive an explicit expression of the stochastic CRB for the DOA parameter alone in the case of non-circular complex Gaussian waveforms observed in additive circular complex Gaussian noise.

2. DATA MODEL

We will be concerned with the signal model

$$\mathbf{z}_t = \mathbf{A}\mathbf{s}_t + \mathbf{n}_t, \qquad t = 1, \dots, T$$

where $(\mathbf{z}_t)_{t=1,\dots,T}$ represents the i.i.d. M-vectors of observed complex envelope at the sensor output. $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_K]$ is the steering matrix where each vector $\mathbf{a}_k$ is parameterized by the real scalar parameter θ_k. $\mathbf{s}_t = (s_{t,1}, \dots, s_{t,K})^T$ and $\mathbf{n}_t$ model signals transmitted by K sources and additive measurement noise respectively. $\mathbf{s}_t$ and $\mathbf{n}_t$ are multivariate independent, complex zero-mean. $\mathbf{n}_t$ is assumed circular complex Gaussian, spatially uncorrelated with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \sigma_n^2\mathbf{I}_M$, while $\mathbf{s}_t$ is non-circular complex Gaussian distributed and possibly spatially correlated or even coherent with $\mathbf{R}_s \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{s}_t\mathbf{s}_t^H)$ and $\mathbf{R}'_s \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{s}_t\mathbf{s}_t^T)$. Consequently this leads to the covariance matrices of $\mathbf{z}_t$:

$$\mathbf{R}_z(\Theta) = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_n^2\mathbf{I}_M \quad \text{and} \quad \mathbf{R}'_z(\Theta) = \mathbf{A}\mathbf{R}'_s\mathbf{A}^T.$$

If no a priori information is available, $(\mathbf{R}_z(\Theta), \mathbf{R}'_z(\Theta))$ is generically parametrized by the $L = K + K^2 + K(K+1) + 1$ real parameters $\Theta = (\Theta_1^T, \Theta_2^T)^T$ with $\Theta_1 \stackrel{\text{def}}{=} (\theta_1, \dots, \theta_K)^T$ and

$$\Theta_2 \stackrel{\text{def}}{=} ((\Re([\mathbf{R}_s]_{i,j}), \Im([\mathbf{R}_s]_{i,j}), \Re([\mathbf{R}'_s]_{i,j}), \Im([\mathbf{R}'_s]_{i,j}))_{1 \le j < i \le K}, ([\mathbf{R}_s]_{i,i}, \Re([\mathbf{R}'_s]_{i,i}), \Im([\mathbf{R}'_s]_{i,i}))_{i=1,\dots,K}, \sigma_n^2)^T.$$

3. INDIRECT DERIVATION OF THE STOCHASTIC CRB FOR NON-CIRCULAR GAUSSIAN SOURCES

To derive the stochastic CRB of the parameter Θ_1 alone, we consider the asymptotic covariance of the ML estimator. We first note that the probability density function of $\mathbf{z}$ considered as a $2M$-variate real Gaussian RV is given by an expression which is similar to that of the PDF in the circular case, provided it is expressed as a function of $\tilde{\mathbf{z}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{z} \\ \mathbf{z}^* \end{pmatrix}$.

$$p(\mathbf{x}, \mathbf{y}) = p'(\tilde{\mathbf{z}}) = (\pi)^{-M}[\mathrm{Det}(\mathbf{R}_{\tilde{z}})]^{-1/2}\exp[-\frac{1}{2}\tilde{\mathbf{z}}^H\mathbf{R}_{\tilde{z}}^{-1}\tilde{\mathbf{z}}]$$

where

$$\mathbf{R}_{\tilde{z}} \stackrel{\text{def}}{=} \mathrm{E}(\tilde{\mathbf{z}}_t\tilde{\mathbf{z}}_t^H) = \tilde{\mathbf{A}}\mathbf{R}_{\tilde{s}}\tilde{\mathbf{A}}^H + \sigma_n^2\mathbf{I}_{2M}$$

with

$$\mathbf{R}_{\tilde{s}} = \begin{bmatrix} \mathbf{R}_s & \mathbf{R}'_s \\ \mathbf{R}'^*_s & \mathbf{R}^*_s \end{bmatrix} \quad (1)$$

and $\tilde{\mathbf{A}} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^* \end{bmatrix}$. Then, classically (see e.g., [5],[1]), after dropping the constants, the log-likelihood function can be written as

$$L(\Theta_1, \Theta_2) = -\frac{T}{2}\left(\ln[\mathrm{Det}(\mathbf{R}_{\tilde{z}})] + \mathrm{Tr}(\mathbf{R}_{\tilde{z}}^{-1}\mathbf{R}_{\tilde{z},T})\right) \quad (2)$$

with $\mathbf{R}_{\tilde{z},T} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T}\tilde{\mathbf{z}}_t\tilde{\mathbf{z}}_t^H$ where the parameters Θ_1 and Θ_2 are imbedded in the covariance matrix $\mathbf{R}_{\tilde{z}}$. In (2), $\mathbf{R}_{\tilde{z}}$ depends on $\mathbf{R}_{\tilde{s}}$, which is structured via (1). Due to these constraints, the ML estimation of (Θ_1, Θ_2) becomes a constrained optimization problem which is not standard. Despite this difficulty, we prove in the following that the ML estimate of the DOA parameters Θ_1 and source and noise covariance parameters Θ_2 may be obtained in a separable form.

Result 1 *If the sample covariance matrix $\mathbf{R}_{\tilde{z},T}$ is positive definite, the joint ML estimates which maximize the log-likelihood function (2) subject to the constraints (1) are given by the following:*
$\widehat{\Theta}_{1,\mathrm{ML}}$ is obtained by the minimizing with respect to Θ_1

$$F_T(\Theta_1) = \ln[\mathrm{Det}(\tilde{\mathbf{A}}\widehat{\mathbf{R}}_{\tilde{s},\mathrm{ML}}\tilde{\mathbf{A}}^H + \widehat{\sigma}^2_{n,\mathrm{ML}}\mathbf{I}_{2M})] \quad (3)$$

where $\widehat{\mathbf{R}}_{\tilde{s},\mathrm{ML}}$ and $\widehat{\sigma}^2_{n,\mathrm{ML}}$ are given by

$$\widehat{\mathbf{R}}_{\tilde{s},\mathrm{ML}} = [\tilde{\mathbf{A}}^H(\Theta_1)\tilde{\mathbf{A}}(\Theta_1)]^{-1}\tilde{\mathbf{A}}^H(\Theta_1) [\mathbf{R}_{\tilde{z},T} - \widehat{\sigma}^2_{n,\mathrm{ML}}\mathbf{I}_{2M}]\tilde{\mathbf{A}}(\Theta_1)[\tilde{\mathbf{A}}^H(\Theta_1)\tilde{\mathbf{A}}(\Theta_1)]^{-1} \quad (4)$$

which is structured as $\mathbf{R}_{\tilde{s}}$ and

$$\widehat{\sigma}^2_{n,\mathrm{ML}} = \frac{1}{M-K}\mathrm{Tr}\left(\mathbf{\Pi}^\perp_{\mathbf{A}(\Theta_1)}\mathbf{R}_{z,T}\right).$$

Proof: Maximizing the log-likelihood (2) without any constraint on the Hermitian matrix $\mathbf{R}_{\tilde{s}}$ reduces to a standard maximization problem. Its solution is given (e.g., in [5], [1]) by the minimization of (3) where $\widehat{\mathbf{R}}_{\tilde{s},\mathrm{ML}}$ is given by (4) and $\widehat{\sigma}^2_{n,\mathrm{ML}}$ by

$$\widehat{\sigma}^2_{n,\mathrm{ML}} = \frac{1}{2M-2K}\mathrm{Tr}\left(\mathbf{\Pi}^\perp_{\tilde{\mathbf{A}}(\Theta_1)}\mathbf{R}_{\tilde{z},T}\right).$$

Because $\mathbf{R}_{\tilde{z},T}$, $\tilde{\mathbf{A}}^H(\Theta_1)\tilde{\mathbf{A}}(\Theta_1)$, then $[\tilde{\mathbf{A}}^H(\Theta_1)\tilde{\mathbf{A}}(\Theta_1)]^{-1}$ and $[\tilde{\mathbf{A}}^H(\Theta_1)\tilde{\mathbf{A}}(\Theta_1)]^{-1}\tilde{\mathbf{A}}^H(\Theta_1)$ are all partitioned of the form $\begin{bmatrix} (\diamond) & (\times) \\ (\times)^* & (\diamond)^* \end{bmatrix}$, the expression (4) is also partitioned of this form. Finally, because $\mathbf{\Pi}_{\tilde{\mathbf{A}}(\Theta_1)} = \begin{bmatrix} \mathbf{\Pi}_{\mathbf{A}(\Theta_1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{\Pi}_{\mathbf{A}(\Theta_1)} \end{bmatrix}$ and $\mathbf{R}_{\tilde{z},T} = \begin{bmatrix} \mathbf{R}_{z,T} & \mathbf{R}'_{z,T} \\ \mathbf{R}'^*_{z,T} & \mathbf{R}^*_{z,T} \end{bmatrix}$ result 1 is proved. ■

Because the dimension of Θ that parametrizes our model is fixed, it follows from the standard statistical theory of ML estimator that the ML estimator of Θ_1 asymptotically (in the number of measurements) achieves the CRB for Θ_1 estimation. Consequently, an explicit expression of the CRB of Θ_1 alone can be derived thanks to an asymptotic analysis of the ML estimate of Θ_1 given by result 1. Thus, by adapting the proof given in [1], the following result is proved in [7].

Result 2 *The normalized (i.e. for $T = 1$) DOA-related block of CRB for non-circular complex Gaussian (NCG) sources is given by the following explicit expression:*

$$\mathbf{C}^{\mathrm{NCG}}_{\Theta_1} = \frac{\sigma_n^2}{2}\left\{\Re\left[\mathbf{D}^H\mathbf{\Pi}^\perp_{\mathbf{A}}\mathbf{D}\odot \left([\mathbf{R}_s\mathbf{A}^H, \mathbf{R}'_s\mathbf{A}^T]\mathbf{R}_{\tilde{z}}^{-1}\begin{bmatrix} \mathbf{A}\mathbf{R}_s \\ \mathbf{A}^*\mathbf{R}'^*_s \end{bmatrix}\right)^T\right]\right\}^{-1} \quad (5)$$

with $\mathbf{D} \stackrel{\text{def}}{=} \frac{d\mathbf{A}(\Theta_1)}{d\Theta_1}$.

Remark: We note that for circular complex Gaussian (CG) sources, $\mathbf{R}'_s = \mathbf{O}$ and $\mathbf{R}_{\tilde{z}} = \begin{bmatrix} \mathbf{R}_z & \mathbf{O} \\ \mathbf{O} & \mathbf{R}^*_z \end{bmatrix}$. Consequently (5) reduces to

$$\mathbf{C}^{\mathrm{CG}}_{\Theta_1} = \frac{\sigma_n^2}{2}\left\{\Re\left[\mathbf{D}^H\mathbf{\Pi}^\perp_{\mathbf{A}}\mathbf{D}\odot\left(\mathbf{R}_s\mathbf{A}^H\mathbf{R}_z^{-1}\mathbf{A}\mathbf{R}_s\right)^T\right]\right\}^{-1}$$

indirectly derived in [1], [2], then directly derived from the circular complex Stepian-Bangs formula in [4].

The next result compares the CRBs $\mathbf{C}^{\mathrm{NCG}}_{\Theta_1}$ and $\mathbf{C}^{\mathrm{CG}}_{\Theta_1}$ associated with sources with the same first covariance matrix $\mathbf{R}_s$.

Result 3 *The DOA-related block of CRB for non-circular complex Gaussian sources is upper bounded by the associated CRB for circular complex Gaussian sources corresponding to the same first covariance matrix $\mathbf{R}_s$.*

$$\mathbf{C}^{\mathrm{NCG}}_{\Theta_1} \leq \mathbf{C}^{\mathrm{CG}}_{\Theta_1}. \quad (6)$$

Proof: First, from [1, lemma A.4], we have $\mathbf{B}_1 - \mathbf{B}_2 \geq \mathbf{O}$ with $\mathbf{B}_1 \stackrel{\text{def}}{=} [\mathbf{R}_s\mathbf{A}^H, \mathbf{R}'_s\mathbf{A}^T]\mathbf{R}_{\tilde{z}}^{-1}\begin{bmatrix} \mathbf{A}\mathbf{R}_s \\ \mathbf{A}^*\mathbf{R}'^*_s \end{bmatrix}$ and

$\mathbf{B}_2 \stackrel{\text{def}}{=} \mathbf{R}_s\mathbf{A}^H\mathbf{R}_z^{-1}\mathbf{A}\mathbf{R}_s$ and this inequality applies to the transpose of these matrices: $\mathbf{B}_1^T - \mathbf{B}_2^T \geq \mathbf{O}$. Then, because $\mathbf{B}_3 \stackrel{\text{def}}{=} \mathbf{D}^H\mathbf{\Pi}_{\mathbf{A}}^{\perp}\mathbf{D} \geq \mathbf{O}$, we have thanks to a standard result of linear algebra (see e.g., [3, Appendix A, result R.19], $\mathbf{B}_3 \odot \left(\mathbf{B}_1^T - \mathbf{B}_2^T\right) \geq \mathbf{O}$. This inequality is extended to the associated real symmetric matrices $\Re\left[\mathbf{B}_3 \odot \mathbf{B}_1^T\right] - \Re\left[\mathbf{B}_3 \odot \mathbf{B}_2^T\right] \geq \mathbf{O}$, then by inversion $\{\Re\left[\mathbf{B}_3 \odot \mathbf{B}_1^T\right]\}^{-1} - \{\Re\left[\mathbf{B}_3 \odot \mathbf{B}_2^T\right]\}^{-1} \leq \mathbf{O}$. ■

In the particular case of one source, we prove the following:

Result 4 *The CRB of θ_1 for a non-circular complex Gaussian source decreases monotonically as the non-circularity rate increases and is given by the expression*

$$C_{\theta_1}^{\mathrm{NCG}} = \frac{1}{\alpha_1}\left[\frac{2r_1^{-1} + ||\mathbf{a}_1||^{-2}r_1^{-2} + ||\mathbf{a}_1||^2 - ||\mathbf{a}_1||^2\rho_1^2}{||\mathbf{a}_1||^2 r_1 + 1 + (1 - ||\mathbf{a}_1||^2 r_1)\rho_1^2}\right] \tag{7}$$

where the non-circularity rate ρ_1 is defined by $\mathrm{E}(s_{t,1}^2) = \rho_1 e^{i\phi_1}\mathrm{E}|s_{t,1}^2|$ and satisfies $0 \leq \rho_1 \leq 1$. The SNR is defined by $r_1 \stackrel{\text{def}}{=} \frac{\sigma_1^2}{\sigma_n^2}$ and α_1 is the purely geometrical factor $2\mathbf{a}_1^{'H}\mathbf{\Pi}_{\mathbf{a}_1}^{\perp}\mathbf{a}_1^{'}$ with $\mathbf{a}_1^{'} \stackrel{\text{def}}{=} \frac{d\mathbf{a}_1}{d\theta_1}$.

Proof: First, note that the structure of the inverse of $\mathbf{R}_{\tilde{z}}$ in (5) is preserved, i.e. $\mathbf{R}_{\tilde{z}}^{-1} = \begin{bmatrix} \mathbf{G} & \mathbf{G}^{'} \\ \mathbf{G}^{'*} & \mathbf{G}^{*} \end{bmatrix}$ with $\mathbf{G} = \left(\mathbf{R}_z - \mathbf{R}_z^{'}\mathbf{R}_z^{*-1}\mathbf{R}_z^{'*}\right)^{-1}$ and $\mathbf{G}^{'} = -\mathbf{G}\mathbf{R}_z^{'}\mathbf{R}_z^{*-1}$. With $\mathbf{R}_z = \sigma_1^2\mathbf{a}_1\mathbf{a}_1^H + \sigma_n^2\mathbf{I}_M$ and $\mathbf{R}_z^{'} = \sigma_1^2\rho_1 e^{i\phi_1}\mathbf{a}_1\mathbf{a}_1^T$, (7) follows thanks to straightforward but tedious calculations. The monotony of $C_{\theta_1}^{\mathrm{NCG}}$ with ρ_1 is proved in [7]. ■

Consequently for one source, the CRB decreases from $C_{\theta_1} = \frac{1}{\alpha_1 r_1}\left(1 + \frac{1}{||\mathbf{a}_1||^2 r_1}\right)$ ($\rho_1 = 0$, circular case) to $C_{\theta_1} = \frac{1}{\alpha_1 r_1}\left(1 + \frac{1}{2||\mathbf{a}_1||^2 r_1}\right)$ ($\rho_1 = 1$, unfiltered BPSK case).

4. DIRECT DERIVATION OF THE STOCHASTIC CRB FOR NON-CIRCULAR GAUSSIAN SOURCES

To directly prove result 2 from the Fisher information matrix, we first extend the circular complex Gaussian Slepian-Bangs formula [3, rel. B.3.25] to non-circular complex Gaussian distribution. This extension is proved in [7] and takes the following form:

$$(\mathbf{I}_F)_{k,l} = \frac{1}{2}\mathrm{Tr}\left[\frac{\partial\mathbf{R}_{\tilde{z}}}{\partial\theta_k}\mathbf{R}_{\tilde{z}}^{-1}\frac{\partial\mathbf{R}_{\tilde{z}}}{\partial\theta_l}\mathbf{R}_{\tilde{z}}^{-1}\right].$$

Second, we note that thanks to the proof of result 1, the constrained ML estimate of Θ_1 coincides with the unconstrained ML estimate of Θ_1. Consequently the associated CRBs of Θ_1 coincide for these two models. Using the unconstrained model, let $\Theta = (\Theta_1^T, \Theta_2^T)^T$ with here $\Theta_2 \stackrel{\text{def}}{=} (\boldsymbol{\rho}^T, \sigma_n^2)^T$ where $\boldsymbol{\rho} \stackrel{\text{def}}{=} ((\Re([\mathbf{R}_{\tilde{s}}]_{i,j}), \Im([\mathbf{R}_{\tilde{s}}]_{i,j}), \Re([\mathbf{R}_{\tilde{s}}^{\prime}]_{i,j}), \Im([\mathbf{R}_{\tilde{s}}^{\prime}]_{i,j}))_{1\leq j<i\leq K}, ([\mathbf{R}_{\tilde{s}}]_{i,i}, \Re([\mathbf{R}_{\tilde{s}}^{\prime}]_{i,i}), \Im([\mathbf{R}_{\tilde{s}}^{\prime}]_{i,i}))_{i=1,\ldots,2K})^T$. With this unconstrained model, we can follow along the lines of the derivation given in [4] where $\mathbf{R}_z = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_n^2\mathbf{I}_M$ is replaced here by $\mathbf{R}_{\tilde{z}} = \tilde{\mathbf{A}}\mathbf{R}_{\tilde{s}}\tilde{\mathbf{A}}^H + \sigma_n^2\mathbf{I}_{2M}$ because the key point of the derivation, i.e. the relation $\mathrm{Vec}(\mathbf{R}_{\tilde{s}}) = \mathbf{J}\boldsymbol{\rho}$ where $\mathbf{J}$ is a constant nonsingular complex matrix is preserved. And rel. (5) is proved in [7] thanks to slight modifications of the direct derivation given in [4].

5. ILLUSTRATIVE EXAMPLES

The purpose of this section is to illustrate results 2, 3 and 4 and to compare these CRBs to the CRB associated with BPSK distributed sources. We consider throughout this section one or two independent and equipowered sources with identical non-circularity rate for which the non-circularity rate ρ_k and the circularity phase ϕ_k are defined by $\mathrm{E}(s_{t,k}^2) = \rho_k e^{i\phi_k}\mathrm{E}|s_{t,k}^2|$. These sources impinge on a uniform linear array of M sensors for which $\mathbf{a}_k = (1, e^{i\theta_k}, \ldots, e^{i(M-1)\theta_k})^T$ with $\theta_k = \pi\sin(\alpha_k)$.

The first experiment illustrates results 2 and 3. We consider two non-circular complex Gaussian sources with $M = 6$ and $SNR = 20dB$. Figs.1, 2 and 3 exhibit the dependence of $\left(\mathbf{C}_{\Theta_1}^{\mathrm{NCG}}\right)_{(1,1)}$ [1] with the non-circularity rate $\rho_1 = \rho_2$, the circularity phase separation $\phi_2 - \phi_1$ and the DOA separation $\theta_2 - \theta_1$, respectively. Fig.1 shows that $\left(\mathbf{C}_{\Theta_1}^{\mathrm{NCG}}\right)_{(1,1)}$ decreases as the non-circularity rate increases (this extends to two equipowered sources result 4 proved in the one source case). Furthermore this decrease is more prominent for low DOA separations. Fig.2 shows that $\left(\mathbf{C}_{\Theta_1}^{\mathrm{NCG}}\right)_{(1,1)}$ is sensitive to the circularity phase separation for low DOA separations. And Fig.3 illustrates the inequality (6) of result 3. It shows that the difference between these two values is very sensitive for very low DOA separations only. Fig.4 compares the non-circular complex Gaussian CRB $\mathbf{C}_{\Theta_1}^{\mathrm{NCG}}$ with the non-circular complex Gaussian CRB $\mathbf{C}_{\Theta_1}^{\mathrm{NCG}'}$ under the a priori information that the two sources are independent [2] given in [6] by a closed-form expression. Fig.4 shows that this a priori information is quite informative, but this information gain decreases as the non-circularity rate increases. This is particularly prominent for low DOA separations.

The second experiment illustrates result 4 where a non-circular complex Gaussian source and $M = 3$ are consid-

[1] All the CRBs are computed for $T = 1$. That means that the actual CRBs associated with the signal model defined in section 2 are obtained from the results given in this section by dividing by T.

[2] We note that the explicit expression (5) does not take account of this a priori information because it has been derived without any constraint on $\mathbf{R}_s$ and $\mathbf{R}_s^{\prime}$.

ered. Fig.5 shows that the CRB decreases monotonically as the non-circularity rate increases but it is relatively insensitive to the increase of ρ_1, except for very low SNR (i.e. for $\|\mathbf{a}_1\|^2 r_1 \approx 1$).

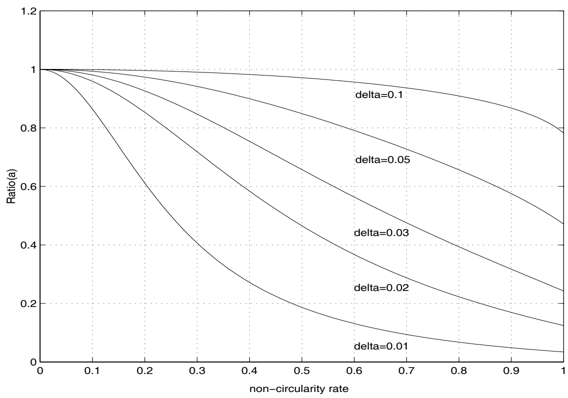

Fig.1 Ratio $r_a \stackrel{\text{def}}{=} \left(\mathbf{C}_{\Theta_1}^{\text{NCG}}\right)_{(1,1)} / \left(\mathbf{C}_{\Theta_1}^{\text{CG}}\right)_{(1,1)}$ as a function of the non-circularity rate for different values of DOA separation (delta) for $\phi_1 = \pi/2$ and $\phi_2 = \pi/3$.

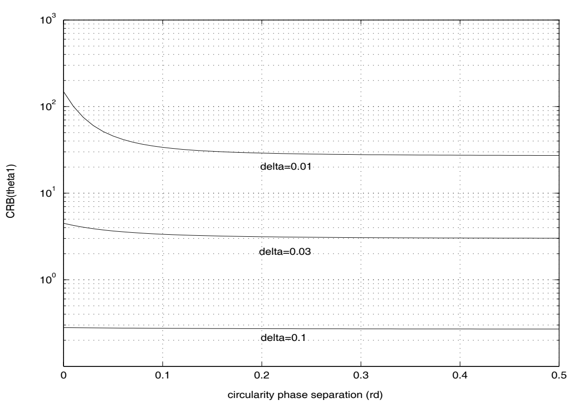

Fig.2 $\left(\mathbf{C}_{\Theta_1}^{\text{NCG}}\right)_{(1,1)}$ as a function of the circularity phase separation for different values of DOA separation (delta) for $\rho_1 = \rho_2 = 1$.

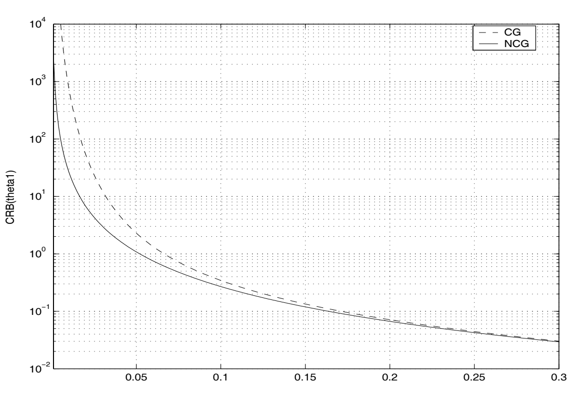

Fig.3 $\left(\mathbf{C}_{\Theta_1}^{\text{NCG}}\right)_{(1,1)}$ and $\left(\mathbf{C}_{\Theta_1}^{\text{CG}}\right)_{(1,1)}$ as a function of the DOA separation for $\rho_1 = \rho_2 = 1$ and $\phi_1 = \pi/2$ and $\phi_2 = \pi/3$.

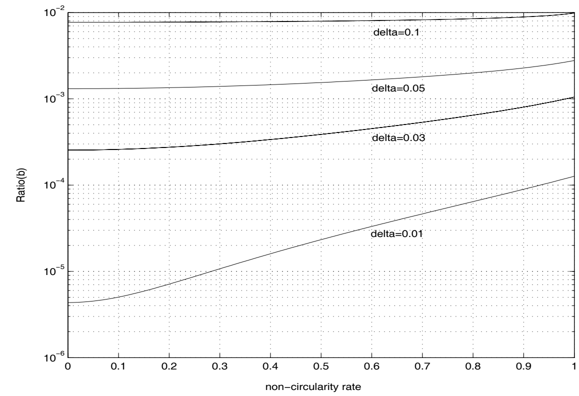

Fig.4 Ratio $r_b \stackrel{\text{def}}{=} \left(\mathbf{C}_{\Theta_1}^{\text{NCG}\prime}\right)_{(1,1)} / \left(\mathbf{C}_{\Theta_1}^{\text{NCG}}\right)_{(1,1)}$ as a function of the non-circularity rate for different values of DOA separation (delta) for $\phi_1 = \pi/2$ and $\phi_2 = \pi/3$.

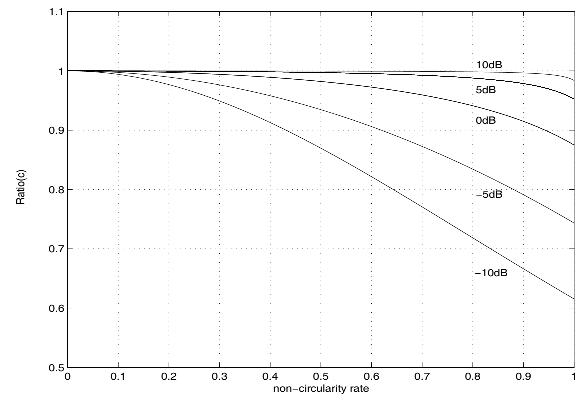

Fig.5 Ratio $r_c \stackrel{\text{def}}{=} \left(C_{\theta_1}^{\text{NCG}}\right) / \left(C_{\theta_1}^{\text{CG}}\right)$ as a function of the non-circularity rate ρ_1 for different values of the SNR r_1.

6. REFERENCES

[1] P. Stoica, A. Nehorai, "Performance study of conditional and unconditional direction of arrival estimation," *IEEE Trans. on Acoustics Speech and Signal Processing*, vol. 38, no. 10, pp. 1783-1795, Oct. 1990.

[2] B. Ottersten, M. Viberg and T. Kailath, "Analysis of subspace fitting and ML techniques for parameter estimation from sensor array data," *IEEE Trans. on Signal Processing*, vol. 40, no. 3, pp. 590-599, March 1992.

[3] P. Stoica, R. Moses, "Introduction to spectral analysis," *Prentice-Hall*, Upper Saddle River, NJ, 1997.

[4] P. Stoica, A.G. Larsson and A.B. Gershman, "The stochastic CRB for array processing: a textbook derivation," *IEEE Signal Proc. letters*, vol. 8, no. 5, pp. 148-150, May 2001.

[5] A.G. Jaffer, "Maximum likelihood direction finding of stochastic sources: a separable solution," *in Proc. ICASSP New York*, pp. 2893-2896, 11-14 April 1988.

[6] J.P. Delmas, "Asymptotically minimum variance second-order estimation for non-circular signal with application to DOA estimation," accepted for publication in *IEEE Trans. on Signal Processing*.

[7] J.P. Delmas, H. Abeida, "Stochastic Cramer-Rao bound for non-circular signals with application to DOA estimation," accepted for publication in *IEEE Trans. on Signal Processing*.