IMPROVING THE ROBUSTNESS OF THE RARE ALGORITHM AGAINST SUBARRAY ORIENTATION ERRORS

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ABSTRACT

We study the problem of direction-of-arrival (DOA) estimation using partly calibrated arrays composed of multiple subarrays with unknown inter-subarray parameters and imperfectly known subarray orientations. The recently developed spectral and root variants of the rank reduction estimator (RARE) can handle scenarios where no calibration between subarrays is available but, unfortunately, they are very sensitive to subarray orientation errors. Therefore, conventional RARE can be applied to such partly calibrated arrays only if all subarray misorientations are negligibly small. In this paper, we develop a new modification of RARE which improves its robustness against subarray misorientations. The performance of the proposed robust RARE algorithm is demonstrated to be close to the stochastic Cramér-Rao bound (CRB) of the considered estimation problem.

1. INTRODUCTION

The problem of DOA estimation in large subarray-based sensor arrays has recently attracted a significant attention of specialists because using subarrays on a sparse grid extends the array aperture without a corresponding increase in hardware and software costs [1]-[5].

In the case when each particular subarray is calibrated but there is no calibration between subarrays (i.e., all inter-subarray parameters are unknown), the recently developed RARE algorithm can be used to estimate the signal DOAs. In the most general setting when all subarrays have arbitrary geometries, the spectral RARE technique [3]-[4] can be used. In the particular case of linear identically oriented subarrays whose interelement spacings are integer multiples of the known shortest baseline, the root-RARE algorithms [5]-[6] can be applied[1].

A serious shortcoming of both the spectral and root-RARE algorithms is that they are very sensitive to array orientation errors. In practical situations, such subarray misorientations may easily occur [7], [8].

In this paper, we develop new spectral and root modifications of the RARE algorithm that are robust against array orientation errors. We also study the identifiability conditions that guarantee the uniqueness of DOA estimates. Our simulation results validate the robustness of the proposed techniques and demonstrate that their performances are close to the stochastic CRB that corresponds to the estimation problem considered.

[1]Note that root-RARE is a search-free algorithm and, because of this, its computational cost is substantially lower than that of spectral RARE.

2. RARE ALGORITHM

Assume that an array of M omnidirectional sensors consists of K arbitrary non-overlapping subarrays. Let the kth subarray have $M_k \geq 1$ sensors, so the total number of sensors in the array is given by $M = \sum_{k=1}^{K} M_k$. Assume that the array receives $L < M$ narrowband signals from multiple far-field sources. In this section, we assume that each subarray is fully calibrated while the inter-subarray displacements may be unknown or uncertain.

The array snapshots can be modeled as [6], [3]

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\alpha})\mathbf{s}(t) + \mathbf{n}(t), \quad t = 1, \cdots, N \tag{1}$$

where

$$\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\alpha}) \triangleq [\mathbf{a}(\theta_1, \boldsymbol{\alpha}), \mathbf{a}(\theta_2, \boldsymbol{\alpha}), \cdots, \mathbf{a}(\theta_L, \boldsymbol{\alpha})] \tag{2}$$

is the $M \times L$ direction matrix, $\mathbf{a}(\theta, \boldsymbol{\alpha})$ is the array steering vector,

$$\boldsymbol{\theta} = [\theta_1, \theta_2, \cdots, \theta_L]^T \tag{3}$$

is the $L \times 1$ vector of the source DOAs, $\boldsymbol{\alpha}$ is the $K \times 1$ vector of unknown inter-subarray parameters, $\mathbf{s}(t)$ is the $L \times 1$ vector of signal waveforms, $\mathbf{n}(t)$ is the $M \times 1$ vector of white circular complex Gaussian noise, N is the number of snapshots, and $(\cdot)^T$ denotes the transpose. The basic idea of the RARE algorithm is to model $\mathbf{a}(\theta, \boldsymbol{\alpha})$ as the product of a known matrix $\mathbf{V}(\theta)$ and an unknown vector $\mathbf{h}(\theta, \boldsymbol{\alpha})$ associated with the unknown inter-subarray parameters [3], [4]

$$\mathbf{a}(\theta, \boldsymbol{\alpha}) = \mathbf{V}(\theta)\mathbf{h}(\theta, \boldsymbol{\alpha}) \tag{4}$$

where

$$\mathbf{V}(\theta) = \begin{bmatrix} \mathbf{v}_1(\theta) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{v}_2(\theta) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{v}_K(\theta) \end{bmatrix} \tag{5}$$

and $\mathbf{v}_k(\theta)$ is the $M_i \times 1$ steering vector of the kth subarray. The unknown vector $\mathbf{h}(\theta, \boldsymbol{\alpha})$ may take different forms, depending on the type of inter-subarray uncertainty considered (see [3]-[5] for details).

Substituting the steering vector model (4) to the MUSIC equation [9]

$$\mathbf{a}^H(\theta, \boldsymbol{\alpha})\mathbf{G}\mathbf{G}^H\mathbf{a}(\theta, \boldsymbol{\alpha}) = 0 \tag{6}$$

we have [3]

$$\mathbf{h}^H(\theta, \boldsymbol{\alpha})\mathbf{C}(\theta)\mathbf{h}(\theta, \boldsymbol{\alpha}) = 0 \tag{7}$$

where $\mathbf{C}(\theta)$ is the $K \times K$ matrix defined as

$$\mathbf{C}(\theta) = \mathbf{V}^H(\theta)\mathbf{G}\mathbf{G}^H\mathbf{V}(\theta) \tag{8}$$

$\mathbf{G}$ is the $M \times (M-L)$ matrix of the noise-subspace eigenvectors of the array covariance matrix $\mathbf{R} = \mathrm{E}\{\mathbf{x}(t)\mathbf{x}^H(t)\}$; and $(\cdot)^H$ denotes the Hermitian transpose.

Since $\mathbf{h}(\theta, \boldsymbol{\alpha}) \neq \mathbf{0}$, equation (7) can hold true only if $\mathbf{C}(\theta)$ drops rank (i.e., $\det\{\mathbf{C}(\theta)\} = 0$) and this property suggests the basic RARE criterion used for DOA estimation. It is worth noting that $\mathbf{C}(\theta)$ does not depend on any of unknown inter-subarray parameters $\boldsymbol{\alpha}$. It has been shown in [6] and [4] that under certain mild conditions, the rank of $\mathbf{C}(\theta)$ drops (i.e., $\mathrm{rank}\{\mathbf{C}(\theta)\} < K$) if and only if θ is equal to one of the source DOAs $\{\theta_i\}_{i=1}^L$.

In practice, $\mathbf{R}$ is unknown and is replaced by its sample estimate

$$\hat{\mathbf{R}} = \frac{1}{N}\sum_{t=1}^{N} \mathbf{x}(t)\mathbf{x}^H(t) \tag{9}$$

In this case, the DOAs are estimated from the L highest peaks of any of the following two alternative estimators [3]-[5]

$$f_1(\theta) = \frac{1}{\det\{\hat{\mathbf{C}}(\theta)\}} \tag{10}$$

$$f_2(\theta) = \frac{1}{\mathcal{L}\{\hat{\mathbf{C}}(\theta)\}} \tag{11}$$

where

$$\hat{\mathbf{C}}(\theta) = \mathbf{V}^H(\theta)\hat{\mathbf{G}}\hat{\mathbf{G}}^H\mathbf{V}(\theta) \tag{12}$$

is the estimate of $\mathbf{C}(\theta)$; $\hat{\mathbf{G}}$ is the $M \times (M-L)$ matrix of the noise-subspace eigenvectors of $\hat{\mathbf{R}}$; and $\mathcal{L}\{\cdot\}$ is the operator returning the smallest eigenvalue of a Hermitian matrix.

The estimators (10) and (11) are based on one-dimensional search over θ. In the particular case of linear identically oriented subarrays whose interelement spacings are integer multiples of the known shortest baseline d, a search-free polynomial rooting-based reformulation of the RARE estimator is possible [6], [5]. In the aforementioned particular case, rewriting the matrix $\hat{\mathbf{C}}(\theta)$ as a function of $z = e^{j\frac{2\pi}{\lambda}d\sin\theta}$ gives

$$\hat{\mathbf{C}}(z) = \mathbf{V}^T(1/z)\hat{\mathbf{G}}\hat{\mathbf{G}}^H\mathbf{V}(z) \tag{13}$$

where λ is the wavelength. Using (13), it can be shown that the DOAs can be obtained by rooting the following polynomial [5]

$$f(z) = \det\{\hat{\mathbf{C}}(z)\} \tag{14}$$

3. ROBUST RARE ALGORITHM

In this section, we assume that subarray orientations are known imprecisely. In this case, the direct application of the RARE algorithm is not possible, as each subarray itself is no longer fully calibrated. In the presence of such subarray orientation errors, the model (4) can be transformed as

$$\mathbf{a}(\theta, \boldsymbol{\zeta}) = \mathbf{V}(\theta, \boldsymbol{\delta_\theta})\mathbf{h}(\theta, \boldsymbol{\zeta}) \tag{15}$$

where $\boldsymbol{\delta_\theta} = [\delta\theta_1, \cdots, \delta\theta_K]^T$; $\delta\theta_k$ is the orientation error of the kth subarray; and $\boldsymbol{\zeta} = [\boldsymbol{\alpha}, \boldsymbol{\delta_\theta}]^T$ is the vector containing all unknown array parameters.

In this case, the $M \times K$ direction matrix $\mathbf{V}(\theta, \boldsymbol{\delta_\theta})$ takes the form

$$\mathbf{V}(\theta, \boldsymbol{\delta_\theta}) = \begin{bmatrix} \mathbf{v}_1(\theta+\delta\theta_1) & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{v}_K(\theta+\delta\theta_K) \end{bmatrix} \tag{16}$$

Assuming small orientation errors, each vector $\mathbf{v}_k(\theta + \delta\theta_k)$ can be expanded using the first two terms of the Taylor series as

$$\mathbf{v}_k(\theta+\delta\theta_k) \simeq \mathbf{v}_k(\theta) + \delta\theta_k \frac{d\mathbf{v}_k(\theta)}{d\theta} \tag{17}$$

Using this approximation, (15) can be written as

$$\mathbf{a}(\theta, \boldsymbol{\zeta}) = \left[\mathbf{V}(\theta), \frac{d\mathbf{V}(\theta)}{d\theta}\right] \begin{bmatrix} \mathbf{h}(\theta, \boldsymbol{\zeta}) \\ \mathbf{Q}\mathbf{h}(\theta, \boldsymbol{\zeta}) \end{bmatrix} \triangleq \mathbf{P}(\theta)\mathbf{g}(\theta, \boldsymbol{\zeta}) \tag{18}$$

where

$$\mathbf{Q} \triangleq \mathrm{diag}\{\delta\theta_1, \cdots, \delta\theta_K\} \tag{19}$$

$$\mathbf{P}(\theta) \triangleq \left[\mathbf{V}(\theta), \frac{d\mathbf{V}(\theta)}{d\theta}\right] \tag{20}$$

$$\mathbf{g}(\theta, \boldsymbol{\zeta}) \triangleq [\mathbf{h}^T(\theta, \boldsymbol{\zeta}), \mathbf{h}^T(\theta, \boldsymbol{\zeta})\mathbf{Q}^T]^T \tag{21}$$

Note that the model (18) is similar to (4) in sense that the matrix $\mathbf{P}(\theta)$ depends only on θ, while all the unknown parameters $\boldsymbol{\zeta}$ are captured in the vector $\mathbf{g}(\theta, \boldsymbol{\zeta})$. However, an important difference between the models (4) and (18) is that (18) describes a more general case when both the inter-subarray parameters and orientation errors are unknown, while (4) corresponds to the case when there are no orientation errors.

The aforementioned similarity between (18) and (4) allows us to use the idea of the conventional RARE algorithm to estimate the source DOAs. In particular, substituting (18) to the MUSIC equation $\mathbf{a}^H(\theta, \boldsymbol{\zeta})\mathbf{G}\mathbf{G}^H\mathbf{a}(\theta, \boldsymbol{\zeta}) = 0$ we have

$$\mathbf{g}^H(\theta, \boldsymbol{\zeta})\mathbf{B}(\theta)\mathbf{g}(\theta, \boldsymbol{\zeta}) = 0 \tag{22}$$

where

$$\mathbf{B}(\theta) = \mathbf{P}^H(\theta)\mathbf{G}\mathbf{G}^H\mathbf{P}(\theta) \tag{23}$$

Since $\mathbf{g}(\theta, \boldsymbol{\zeta}) \neq \mathbf{0}$, (22) can hold true only if the matrix $\mathbf{B}(\theta)$ drops rank. Therefore, to estimate the signal DOAs in the finite sample case, we can use the following spectral functions similar to (10) and (11):

$$\tilde{f}_1(\theta) = \frac{1}{\det\{\hat{\mathbf{B}}(\theta)\}} \tag{24}$$

$$\tilde{f}_2(\theta) = \frac{1}{\mathcal{L}\{\hat{\mathbf{B}}(\theta)\}} \tag{25}$$

where

$$\hat{\mathbf{B}}(\theta) = \mathbf{P}^H(\theta)\hat{\mathbf{G}}\hat{\mathbf{G}}^H\mathbf{P}(\theta) \tag{26}$$

In the specific case of linear subarrays whose inter-element spacings are integer multiples of the shortest baseline d, we will use the same approach to reformulate robust RARE in a search-free form. Using (18), we can rewrite the steering vector as

$$\begin{aligned} \mathbf{a}(\theta, \boldsymbol{\zeta}) &= \left[\mathbf{V}(z), \frac{d\mathbf{V}(z)}{d\theta}\right] \begin{bmatrix} \mathbf{h}(\theta, \boldsymbol{\zeta}) \\ \mathbf{Q}\mathbf{h}(\theta, \boldsymbol{\zeta}) \end{bmatrix} \\ &= \left[\mathbf{V}(z), \frac{d\mathbf{V}(z)}{dz}\right] \begin{bmatrix} \mathbf{h}(\theta, \boldsymbol{\zeta}) \\ u(\theta)\mathbf{Q}\mathbf{h}(\theta, \boldsymbol{\zeta}) \end{bmatrix} \\ &\triangleq \mathbf{F}(z)\mathbf{p}(\theta, \boldsymbol{\zeta}) \end{aligned} \tag{27}$$

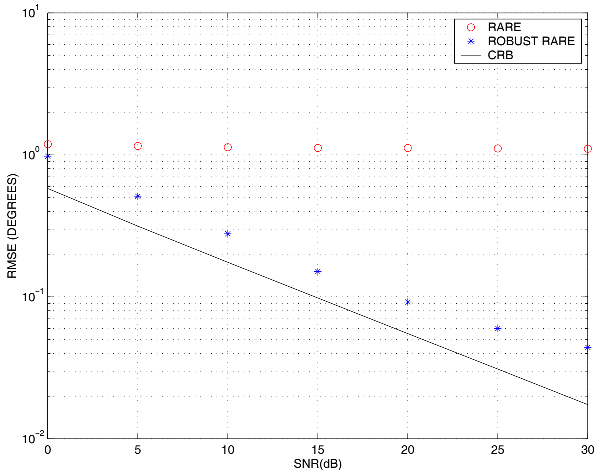

Figure 1: RMSEs versus SNR. First example.

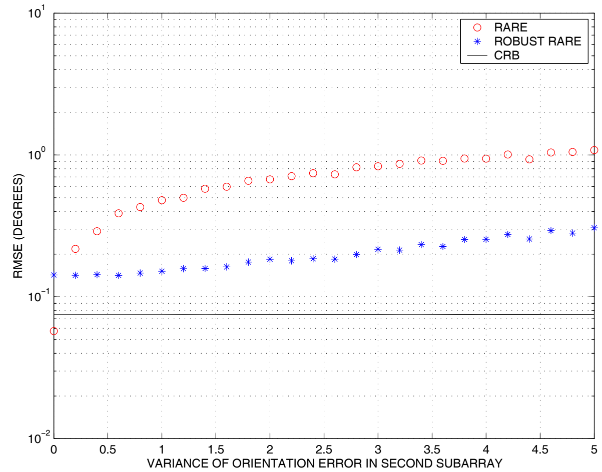

Figure 2: RMSEs versus σ_v^2. Second example.

where

$$\mathbf{F}(z) \triangleq \left[\mathbf{V}(z), \frac{d\mathbf{V}(z)}{dz}\right] \quad (28)$$

$$\mathbf{p}(\theta, \boldsymbol{\zeta}) \triangleq [\mathbf{h}^T(\theta, \boldsymbol{\zeta}), u(\theta)\mathbf{h}^T(\theta, \boldsymbol{\zeta})\mathbf{Q}^T]^T \quad (29)$$

and $u(\theta) = j\frac{2\pi}{\lambda} d\cos\theta$. In (27), we have taken into account that

$$\frac{d\mathbf{v}_k(\theta)}{d\theta} = \frac{d\mathbf{v}_k(\theta)}{dz}\frac{dz}{d\theta} \quad (30)$$

Note that the matrix $\mathbf{F}(z)$ is a function of z only. This allows us to estimate the signal DOAs by means of rooting the polynomial

$$\tilde{f}(z) = \det\{\hat{\mathbf{E}}(z)\} \quad (31)$$

where

$$\hat{\mathbf{E}}(z) = \mathbf{F}^H(z)\hat{\mathbf{G}}\hat{\mathbf{G}}^H\mathbf{F}(z) \quad (32)$$

4. IDENTIFIABILITY AND UNIQUENESS

In this section, we study the identifiability conditions for our robust RARE algorithms.

Following [6]-[5], an array (or a subarray) is said to be *unambiguous* if its steering vectors at any two distinct DOAs are linearly independent. According to the classification given in [10], this is a *trivial ambiguity* which means that an array (or a subarray) does not have grating lobes. Note that our definition of unambiguous arrays (subarrays) is based on the consideration of trivial ambiguities only. In this case, the uniqueness of the DOA estimates can be established in the *almost sure* sense [5].

The following theorem establishes the almost sure uniqueness of the robust RARE DOA estimates in the infinite sample case:

Theorem: Let

$$\beta_k = \begin{cases} 1\,, & \mathcal{M}_k \text{ and } \tilde{\mathcal{M}}_k \text{ are unambiguous} \\ 0\,, & \text{otherwise} \end{cases} \quad (33)$$

where $\mathcal{M}_k$ and $\tilde{\mathcal{M}}_k$ are the manifolds that correspond to the steering vector $\mathbf{v}_k(\theta)$ and its derivative $\mathbf{v}_k(\theta)/d\theta$, respectively.

If the following condition is satisfied

$$L \leq \sum_{k=1}^{K} \beta_k(M_k - 2) \quad (34)$$

then the matrix $\mathbf{B}(\theta)$ drops rank if and only if $\theta \in \{\theta_1, \cdots, \theta_L\}$. $\square$

Proof: See [11].

It is worth noting that the identifiability condition (34) is more strict than the condition obtained in [4] and [6] for the conventional RARE estimator.

5. SIMULATION RESULTS

In our simulations, we assume an array composed of two subarrays and two uncorrelated sources with the DOAs $\theta_1 = 10^\circ$ and $\theta_2 = 20^\circ$. The inter-subarray displacement vector is $[1.3\lambda, 1.2\lambda]^T$. The first subarray is free of orientation errors, while the second subarray suffers from an orientation error. This error is either constant (examples 1 and 3) or random (examples 2 and 4). The number of snapshots is $N = 100$, and all results are averaged over 200 simulation runs. In each figure, the related stochastic CRB is also displayed. The latter bound has been derived in [11].

In the first example, we assume that the first subarray consists of six sensors at the locations $\{(0,0), (0.4\lambda, 0.2\lambda), (0.9\lambda, 0.4\lambda), (1.4\lambda, 0.7\lambda), (1.7\lambda, 1.1\lambda), (2.1\lambda, 1.3\lambda)\}$, while the second subarray has six sensors at the locations $\{(0,0), (0.4\lambda, 0.3\lambda), (0.8\lambda, 0.5\lambda), (1.3\lambda, 0.7\lambda), (1.9\lambda, \lambda), (2.3\lambda, 1.3\lambda)\}$. Note here that the sensor locations for each subarray are indicated relative to its first sensor. The second subarray has an orientation error of 2°. The conventional and robust spectral RARE algorithms are compared in Figure 1 which shows the DOA estimation root-mean-square errors (RMSEs) versus the signal-to-noise ratio (SNR).

In the second example, we assume that the first subarray consists of four sensors located at $\{(0,0), (0.3\lambda, 0.4\lambda), (0.7\lambda, 0.8\lambda), (1.2\lambda, 1.1\lambda)\}$, while the second subarray has six sensors located at $\{(0,0), (0.4\lambda, 0.6\lambda), (0.91\lambda, \lambda), (1.3\lambda, 1.3\lambda), (1.6\lambda, 1.7\lambda), (1.9\lambda, 2\lambda)\}$. The second subarray suffers from a random orientation error which changes from run to run and is assumed to have Gaussian distribution with zero mean and variance σ_v^2. In Figure

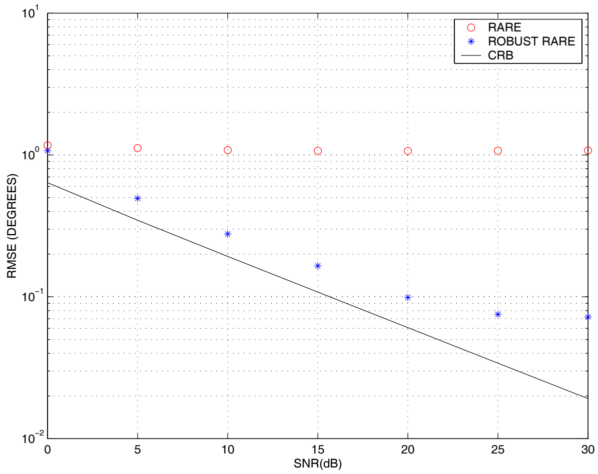

Figure 3: RMSEs versus SNR. Third example.

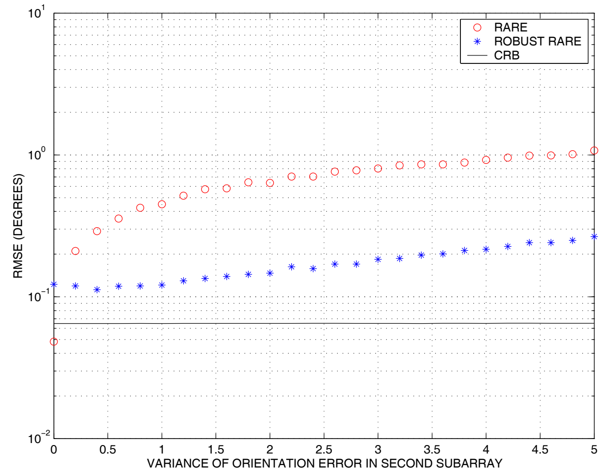

Figure 4: RMSEs versus σ_v^2. Fourth example.

2, we plot the DOA estimation RMSEs of the conventional and robust spectral RARE algorithms versus σ_v^2 at SNR= 20 dB.

In our third example, the first subarray is a ULA of four sensors with the interelement spacing of 0.5λ, and the second subarray is a ULA of six sensors with the interelement spacing of 0.4λ. The first subarray does not suffer from any orientation error while the second subarray has an orientation error of $2°$. The conventional and robust root-RARE algorithms are compared in Figure 3 which displays the DOA estimation RMSEs versus the SNR.

In the last example, both subarrays are assumed to have the same configuration as in the previous example. The second subarray has a random Gaussian orientation error with zero mean and variance σ_v^2. Figure 4 illustrates the DOA estimation RMSEs of the conventional and robust root-RARE algorithms versus σ_v^2 at SNR= 20 dB.

All figures show that in the presence of subarray orientation errors, the robust RARE algorithms have substantially better performance than the conventional RARE techniques. In fact, the RMSE of robust RARE remains close enough to the corresponding CRB, whereas the RMSE of conventional RARE may be far away from this bound (Figures 1 and 2).

6. CONCLUSIONS

The problem of DOA estimation using partly calibrated arrays composed of multiple subarrays with unknown inter-subarray parameters and imperfectly known subarray orientations has been studied. The robust modifications of the RARE algorithm have been proposed that improve its DOA estimation performance in the case of subarray orientation errors. Simulation results validate substantial performance improvements achieved by the new DOA estimation techniques. In particular, the performance of the proposed robust RARE algorithm has been shown to be close to the stochastic CRB of the considered estimation problem.

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